

Statistics for Poisson models of overlapping spheres

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Abstract

The paper considers the stationary Poisson Boolean model with spherical grains and proposes a family of nonparametric estimators for the radius distribution. These estimators are based on observed distances and radii, weighted in an appropriate way. They are ratio-unbiased and asymptotically consistent for growing observation window. It is shown that the asymptotic variance exists and is given by a fairly explicit integral expression. Asymptotic normality is established under a suitable integrability assumption on the weight function. The paper also provides a short discussion of related estimators as well as a simulation study.

1 Introduction

We consider a stationary random closed set Z in \mathbb{R}^d ($d \geq 2$) which is given as a union of random balls of the form

$$Z := \bigcup_{n \geq 1} B(\xi_n, R_n), \quad (1.1)$$

where $B(x, r)$ is a closed Euclidean ball with radius $r \geq 0$ centered at $x \in \mathbb{R}^d$, $\Phi := \{\xi_n : n \geq 1\}$ is a stationary Poisson point process on \mathbb{R}^d , and the sequence $(R_n)_{n \geq 1}$ is independent of Φ and is formed by independent non-negative random variables with common distribution \mathbb{G} . Let R be a generic random variable with distribution \mathbb{G} . We will always assume that it has a finite $2d$ -th moment, that is,

$$\mathbb{E}R^{2d} < \infty. \quad (1.2)$$

Definition (1.1) provides an important model in stochastic geometry with numerous applications in physics and materials science, for instance. The set Z is called a *stationary Boolean model with spherical grains*. A simulated realization for $d = 2$ is shown in Figure 1.

It is a fundamental statistical problem to retrieve information on \mathbb{G} based on an observation of Z in a bounded window W . Our aim in this paper is to propose and study a family of nonparametric estimators of \mathbb{G} . The nonparametric estimation of the radius distribution \mathbb{G} has been studied before; see [3, Chapter 5.6], [14] or [16]. In [16] a kernel estimator is obtained by the method of tangent points. The asymptotic properties of this estimator are studied in [8]. For earlier work on statistics for the Boolean model, we refer to [19, Chapter 3.3], [15] and the references therein.

In the following, we assume that all random elements are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a Borel set $A \subset \mathbb{R}^d$, we write $\Phi(A) := \text{card}\{n \geq 1 : \xi_n \in A\}$ and assume that Φ has a positive and finite intensity

$$\gamma := \mathbb{E}\Phi([0, 1]^d).$$

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Throughout the paper, let B be a compact convex set which contains the origin \mathbf{o} and a non-degenerate segment. We call B *structuring element* or *gauge body*, but we point out that B need not be centrally symmetric or full-dimensional. The B -distance from a point $x \in \mathbb{R}^d$ to a set $A \subset \mathbb{R}^d$ is

$$d_B(x, A) := \inf\{r \geq 0 : (x + rB) \cap A \neq \emptyset\} \in [0, \infty].$$

Clearly, if $\mathbf{o} \in \text{int } B$ and A is nonempty and closed, then the infimum is a minimum. The most common structuring element is the unit ball $B(\mathbf{o}, 1)$, for which we also write B^d and which is based on a scalar product and a norm denoted by $\|\cdot\|$. For given $x \notin Z$, almost surely $d_B(x, Z) < \infty$ whenever R satisfies $\mathbb{P}(R > 0) > 0$. We always assume that this condition is fulfilled. Then almost surely there is a unique $n \in \mathbb{N}$ (that is, a ball $B(\xi_n, R_n)$) such that $(x + d_B(x, Z)B) \cap B(\xi_n, R_n) \neq \emptyset$ (see [10, Lemma 3.1] or [20, Lemma 9.5.1]). In this case, we define $r_B(x, Z)$ as R_n . Figure 1 illustrates the definition of $d_B(x, Z)$ and $r_B(x, Z)$ for $x \notin Z$ and $B = B^2$.

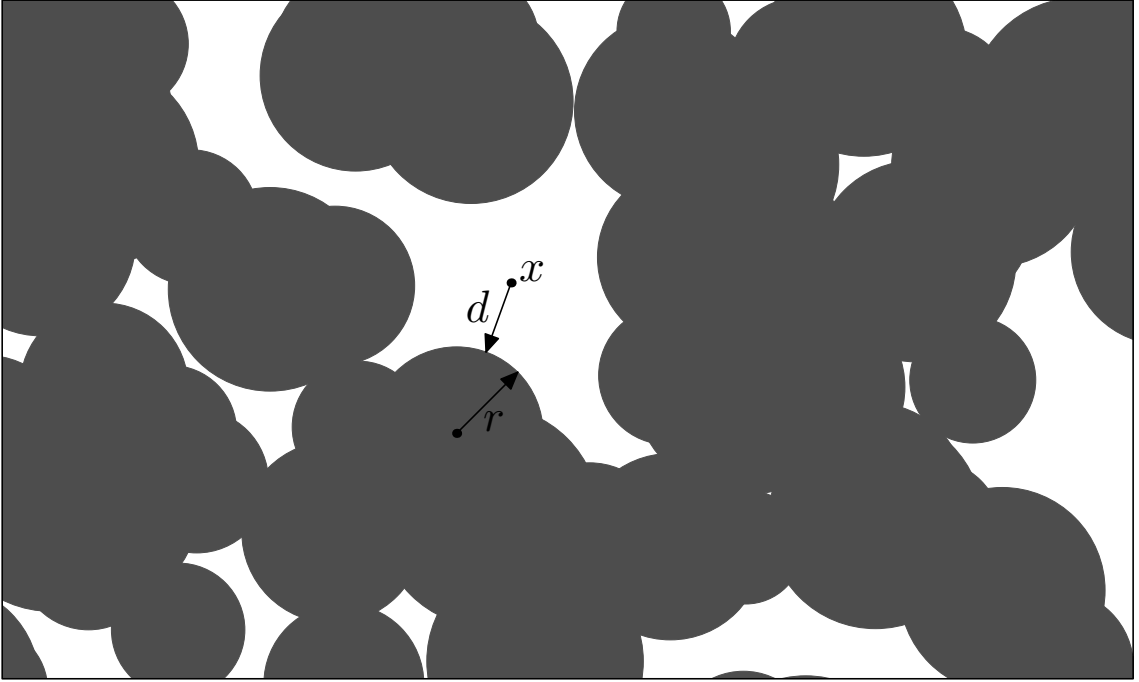


Figure 1: A simulated realization of a planar stationary Boolean model Z with spherical grains observed in a rectangular observation window. The symbol d denotes $d_{B^2}(x, Z)$ and r stands for $r_{B^2}(x, Z)$.

For $s, r \geq 0$, we write $B_{s,r} := sB \oplus rB^d$ for the Minkowski sum of sB and rB^d . Let $|A|_d$ denote the d -dimensional Lebesgue measure of a set $A \subset \mathbb{R}^d$, let $\kappa_k := |B^k|_k = \pi^{k/2}/\Gamma(1 + k/2)$ denote the volume of the k -dimensional unit ball and write $V_j(B)$ for the j -th intrinsic volume of B (see [20, Chapter 14.3]). Then, for $t \in \mathbb{R}^+ := [0, \infty)$, the *empty space function* F_B of Z is given by

$$\begin{aligned} F_B(t) &:= \mathbb{P}(d_B(\mathbf{o}, Z) \leq t) = \mathbb{P}(Z \cap tB \neq \emptyset) = 1 - \exp\{-\gamma \mathbb{E}|B_{t,R}|_d\} \\ &= 1 - \exp\left\{-\gamma \sum_{j=0}^d \kappa_{d-j} V_j(B) t^j \mathbb{E} R^{d-j}\right\}. \end{aligned} \quad (1.3)$$

The empty space function is a useful summary statistics of random sets (see [19, 5]). In the case of a strictly convex gauge body B a detailed study of F_B for (non-stationary) germ-grain models can be found in [9]. We denote the complementary empty space function by $\bar{F}_B(t) := 1 - F_B(t)$. As a

consequence of [10, Theorem 3.2], we get for all measurable functions $\tilde{g} : [0, \infty] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\tilde{g}(0, r) = \tilde{g}(\infty, r) = 0$, $r \in \mathbb{R}^+$, and all $x \in \mathbb{R}^d$ that

$$\mathbb{E}\tilde{g}(d_B(x, Z), r_B(x, Z)) = \gamma \int_0^\infty \int_0^\infty \tilde{g}(t, r) h_B(t, r) \bar{F}_B(t) dt \mathbb{G}(dr) \quad (1.4)$$

with

$$h_B(t, r) := \sum_{j=0}^{d-1} (j+1) \kappa_{d-1-j} V_{j+1}(B) r^{d-1-j} t^j,$$

for $t, r \in [0, \infty)$; see also [20, Theorem 9.5.2]. Note that on the left-hand side of (1.4) the restriction to $\{0 < d_B(x, Z) < \infty\}$ is expressed by the condition $\tilde{g}(0, r) = \tilde{g}(\infty, r) = 0$.

For Borel sets $C \subset \mathbb{R}^+$, $A \subset \mathbb{R}^d$ and a measurable function $f : [0, \infty] \rightarrow \mathbb{R}^+$ with $f(0) = f(\infty) = 0$, we define a random measure η_A by

$$\eta_A(C) := \int_A \mathbf{1}\{r_B(x, Z) \in C\} f(d_B(x, Z)) h_B(d_B(x, Z), r_B(x, Z))^{-1} dx, \quad (1.5)$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Here we put $0/0 := 0$. Thus, in particular, the integration effectively extends over the complement $Z^c := \{x \in \mathbb{R}^d : d_B(x, Z) > 0\}$ of Z . Throughout the paper we shall assume that

$$0 < \beta := \int_0^\infty f(t) \bar{F}_B(t) dt < \infty. \quad (1.6)$$

In view of (1.3) this is a rather weak assumption on f . Moreover, we assume that the origin is an interior point of B if $\mathbb{P}(R = 0) > 0$. This assumption ensures that $h_B(t, r) > 0$ for $t \in (0, \infty)$ and \mathbb{G} -almost all $r \in \mathbb{R}^+$. By Fubini's theorem and (1.4), we obtain

$$\mathbb{E}\eta_A(C) = \gamma \beta |A|_d \mathbb{G}(C). \quad (1.7)$$

Consider a compact convex observation window $W \subset \mathbb{R}^d$ with $|W|_d > 0$. We propose an estimator $\hat{\mathbb{G}}$ for \mathbb{G} based on the information contained in the data $\{(d_B(x, Z), r_B(x, Z)) : x \in W \setminus Z\}$. Note that these data may also require information from outside W . The estimator is given by

$$\hat{\mathbb{G}}(C) := \frac{\eta_W(C)}{\eta_W(\mathbb{R}^+)}, \quad (1.8)$$

where $C \subset \mathbb{R}^+$ is a Borel set. If the denominator in (1.8) is zero, then the numerator is zero as well, and we use the convention $0/0 := 0$. From (1.7) we see that $\mathbb{E}\eta_W(C) = \gamma \beta |W|_d \mathbb{G}(C)$ and $\mathbb{E}\eta_W(\mathbb{R}^+) = \gamma \beta |W|_d$. This means that $\hat{\mathbb{G}}$ is a *ratio-unbiased* estimator of \mathbb{G} .

The paper is organized as follows. In Section 2 we study second order properties of (1.5). We show that the asymptotic variance exists and is given by a fairly explicit integral expression. Consequently, the estimator (1.8) is asymptotically weakly consistent as the compact convex observation window W is expanding. Strong consistency follows from the spatial ergodic theorem. Section 3 contains the proof of asymptotic normality under an integrability assumption on the function f . In Section 4 we consider the estimator $\hat{\mathbb{G}}$ in the plane and for the spherical ($B = B(o, 1)$) as well as for the linear case (B a segment). We also discuss some related estimators. A simulation study is performed to compare the behaviour of different discrete versions of these estimators of the radius distribution \mathbb{G} .

2 Second order properties

For a Borel set $A \subset \mathbb{R}^d$, we define the *restricted Boolean model* as

$$Z(A) := \bigcup_{n: \xi_n \in A} B(\xi_n, R_n).$$

Clearly, $Z(A)$ is not stationary unless $A = \mathbb{R}^d$. Further, for $t \in \mathbb{R}^+$ the complementary empty space function of $Z(A)$ with respect to $x \in \mathbb{R}^d$ is defined by

$$\begin{aligned} \bar{F}_B^A(x; t) &:= \mathbb{P}(d_B(x, Z(A)) > t) = \mathbb{P}((x + tB) \cap Z(A) = \emptyset) \\ &= \mathbb{E} \prod_{n \geq 1} (1 - \mathbf{1}\{(x + tB) \cap B(\xi_n, R_n) \neq \emptyset\} \mathbf{1}\{\xi_n \in A\}) \\ &= \exp \left\{ -\gamma \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}\{(x + tB) \cap B(y, R) \neq \emptyset\} \mathbf{1}\{y \in A\} dy \right\} \\ &= \exp \left\{ -\gamma \mathbb{E} |(x + B_{t,R}) \cap A|_d \right\}. \end{aligned} \quad (2.1)$$

In particular, we have $\bar{F}_B^{\mathbb{R}^d}(x; t) = \bar{F}_B(t)$.

For Borel sets $A_1, A_2 \subset \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$, it will be convenient to introduce the complementary *second-order empty space function* with respect to $x_1, x_2 \in \mathbb{R}^d$ as

$$\bar{F}_B^{A_1, A_2}(x_1, x_2; t_1, t_2) := \mathbb{P}(d_B(x_1, Z(A_1)) > t_1, d_B(x_2, Z(A_2)) > t_2) \quad (2.2)$$

$$\begin{aligned} &= \mathbb{P}((x_1 + t_1 B) \cap Z(A_1) = \emptyset, (x_2 + t_2 B) \cap Z(A_2) = \emptyset) \\ &= \mathbb{E} \prod_{n \geq 1} (1 - \mathbf{1}\{(x_1 + t_1 B) \cap B(\xi_n, R_n) \neq \emptyset\} \mathbf{1}\{\xi_n \in A_1\}) \\ &\quad \times (1 - \mathbf{1}\{(x_2 + t_2 B) \cap B(\xi_n, R_n) \neq \emptyset\} \mathbf{1}\{\xi_n \in A_2\}) \\ &= \exp \left\{ -\gamma \mathbb{E} |((x_1 + B_{t_1, R}) \cap A_1) \cup ((x_2 + B_{t_2, R}) \cap A_2)|_d \right\}. \end{aligned} \quad (2.3)$$

This function is related to the second-order contact distribution function which is studied in [1].

In order to obtain a more concise statement in the subsequent Lemma 2.1 (and again in the proof of Theorem 3.1), we introduce for given Borel sets $A_1, A_2 \subset \mathbb{R}^d$ two functions, $I_1(A_1, A_2)$ and $I_2(A_1, A_2)$, depending on the arguments $(x_1, x_2, y, r) \in (\mathbb{R}^d)^3 \times \mathbb{R}^+$ and $(x_1, x_2, y_1, y_2, r_1, r_2) \in (\mathbb{R}^d)^4 \times (\mathbb{R}^+)^2$, respectively, which are defined by

$$I_1(A_1, A_2)(x_1, x_2, y, r) := \mathbf{1}\{y \in A_1 \cap A_2\} \bar{F}_B^{A_1, A_2}(x_1, x_2; d_B(x_1, B(y, r)), d_B(x_2, B(y, r)))$$

and

$$\begin{aligned} I_2(A_1, A_2)(x_1, x_2, y_1, y_2, r_1, r_2) &:= \mathbf{1}\{y_1 \in A_1\} \mathbf{1}\{y_2 \in A_2\} \\ &\times \left[\left(1 - \mathbf{1}\{y_2 \in A_1\} \mathbf{1}\{d_B(x_1, B(y_2, r_2)) \leq d_B(x_1, B(y_1, r_1))\} \right) \right. \\ &\times \left(1 - \mathbf{1}\{y_1 \in A_2\} \mathbf{1}\{d_B(x_2, B(y_1, r_1)) \leq d_B(x_2, B(y_2, r_2))\} \right) \\ &\times \bar{F}_B^{A_1, A_2}(x_1, x_2; d_B(x_1, B(y_1, r_1)), d_B(x_2, B(y_2, r_2))) \\ &\quad \left. - \bar{F}_B^{A_1}(x_1; d_B(x_1, B(y_1, r_1))) \bar{F}_B^{A_2}(x_2; d_B(x_2, B(y_2, r_2))) \right]. \end{aligned}$$

If the arguments of these two functions are clear from the context, they are sometimes omitted.

Lemma 2.1. *Let $A_1, A_2 \subset \mathbb{R}^d$ be Borel sets and let $x_1, x_2 \in \mathbb{R}^d$. If $\tilde{g} : [0, \infty] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a measurable function with $\tilde{g}(0, r) = \tilde{g}(\infty, r) = 0$ for $r \in \mathbb{R}^+$, then*

$$\begin{aligned} & \text{Cov}\left(\tilde{g}(d_B(x_1, Z(A_1))), r_B(x_1, Z(A_1))), \tilde{g}(d_B(x_2, Z(A_2))), r_B(x_2, Z(A_2))\right) \\ &= \gamma \int_0^\infty \int_{\mathbb{R}^d} \tilde{g}(d_B(x_1, B(y, r)), r) \tilde{g}(d_B(x_2, B(y, r)), r) I_1(A_1, A_2)(x_1, x_2, y, r) dy \mathbb{G}(dr) \\ &+ \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1) \tilde{g}(d_B(x_2, B(y_2, r_2)), r_2) \\ &\quad \times I_2(A_1, A_2)(x_1, x_2, y_1, y_2, r_1, r_2) dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2). \end{aligned}$$

Proof. For $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $i \in \{1, 2\}$, we define the event

$$D_n^{(i)}(x) := \left\{ d_B(x, \bigcup_{k \neq n: \xi_k \in A_i} B(\xi_k, R_k)) > d_B(x, B(\xi_n, R_n)) \right\}.$$

Then

$$\begin{aligned} & \mathbb{E} \tilde{g}(d_B(x_1, Z(A_1))), r_B(x_1, Z(A_1))) \tilde{g}(d_B(x_2, Z(A_2))), r_B(x_2, Z(A_2))) \\ &= \mathbb{E} \sum_{n: \xi_n \in A_1} \sum_{m: \xi_m \in A_2} \mathbf{1}_{D_n^{(1)}(x_1) \cap D_m^{(2)}(x_2)} \tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \tilde{g}(d_B(x_2, B(\xi_m, R_m)), R_m) \\ &= \mathbb{E} \sum_{n: \xi_n \in A_1 \cap A_2} \mathbf{1}_{D_n^{(1)}(x_1) \cap D_n^{(2)}(x_2)} \tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \tilde{g}(d_B(x_2, B(\xi_n, R_n)), R_n) \\ &\quad + \mathbb{E} \sum_{n \neq m: \xi_n \in A_1, \xi_m \in A_2} \mathbf{1}_{D_n^{(1)}(x_1) \cap D_m^{(2)}(x_2)} \tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \tilde{g}(d_B(x_2, B(\xi_m, R_m)), R_m) \\ &=: J_1 + J_2. \end{aligned}$$

Applying Mecke's formula (see [20, Corollary 3.2.3]), we obtain

$$\begin{aligned} J_1 &= \gamma \int_0^\infty \int_{A_1 \cap A_2} \tilde{g}(d_B(x_1, B(y, r)), r) \tilde{g}(d_B(x_2, B(y, r)), r) \\ &\quad \times \mathbb{P}\left(d_B(x_1, Z(A_1)) > d_B(x_1, B(y, r)), d_B(x_2, Z(A_2)) > d_B(x_2, B(y, r))\right) dy \mathbb{G}(dr) \end{aligned}$$

and

$$\begin{aligned} J_2 &= \gamma^2 \int_0^\infty \int_0^\infty \int_{A_2} \int_{A_1} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1) \tilde{g}(d_B(x_2, B(y_2, r_2)), r_2) \\ &\quad \times \mathbb{E} \mathbf{1}\{d_B(x_1, Z_{y_2}(A_1)) > d_B(x_1, B(y_1, r_1))\} \mathbf{1}\{d_B(x_2, Z_{y_1}(A_2)) > d_B(x_2, B(y_2, r_2))\} \\ &\quad \times dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2), \end{aligned}$$

where $Z_{y_2}(A_1) = Z(A_1) \cup B(y_2, r_2)$ if $y_2 \in A_1$ and $Z_{y_2}(A_1) = Z(A_1)$ if $y_2 \notin A_1$. Analogously,

$Z_{y_1}(A_2) = Z(A_2) \cup B(y_1, r_1)$ if $y_1 \in A_2$ and $Z_{y_1}(A_2) = Z(A_2)$ if $y_1 \notin A_2$. Hence,

$$\begin{aligned}
J_2 &= \gamma^2 \int_0^\infty \int_0^\infty \int_{A_2} \int_{A_1} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1) \tilde{g}(d_B(x_2, B(y_2, r_2)), r_2) \\
&\quad \times \left(1 - \mathbf{1}\{y_2 \in A_1\} \mathbf{1}\{d_B(x_1, B(y_2, r_2)) \leq d_B(x_1, B(y_1, r_1))\}\right) \\
&\quad \times \left(1 - \mathbf{1}\{y_1 \in A_2\} \mathbf{1}\{d_B(x_2, B(y_1, r_1)) \leq d_B(x_2, B(y_2, r_2))\}\right) \\
&\quad \times \mathbb{P}\left(d_B(x_1, Z(A_1)) > d_B(x_1, B(y_1, r_1)), d_B(x_2, Z(A_2)) > d_B(x_2, B(y_2, r_2))\right) \\
&\quad \times dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbb{E} \tilde{g}(d_B(x_1, Z(A_1)), r_B(x_1, Z(A_1))) &= \mathbb{E} \sum_{n: \xi_n \in A_1} \mathbf{1}_{D_n^{(1)}(x_1)} \tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \\
&= \gamma \int_0^\infty \int_{A_1} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1) \bar{F}_B^{A_1}(x_1; d_B(x_1, B(y_1, r_1))) dy_1 \mathbb{G}(dr_1).
\end{aligned}$$

□

Our aim is to analyze the second-order properties of the random measure η_A given by (1.5). For this reason, we work with the complementary second-order empty space function (2.2). For $A_1 = A_2 = \mathbb{R}^d$, $t_1, t_2 \in \mathbb{R}^+$, and $u = x_2 - x_1$, by the stationarity of Z this function turns into

$$\begin{aligned}
\bar{F}_B^{(2)}(u; t_1, t_2) &:= \mathbb{P}(d_B(o, Z) > t_1, d_B(u, Z) > t_2) \\
&= \exp\left\{-\gamma \mathbb{E}|B_{t_1, R} \cup (u + B_{t_2, R})|_d\right\} \\
&= \bar{F}_B(t_1) \bar{F}_B(t_2) \exp\{\gamma \mathbb{E} \kappa_B(u; t_1, t_2, R)\},
\end{aligned} \tag{2.4}$$

where

$$\kappa_B(u; t_1, t_2, r) := |B_{t_1, r} \cap (u + B_{t_2, r})|_d. \tag{2.5}$$

Observe that for any $u \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$, we have

$$\bar{F}_B^{(2)}(u; t_1, t_2) \geq \bar{F}_B(t_1) \bar{F}_B(t_2) \tag{2.6}$$

and

$$\bar{F}_B^{(2)}(u; t_1, t_2) \leq \exp\left\{-\frac{\gamma}{2} \mathbb{E}(|B_{t_1, R}|_d + |B_{t_2, R}|_d)\right\} = \sqrt{\bar{F}_B(t_1) \bar{F}_B(t_2)}. \tag{2.7}$$

These inequalities will be used subsequently. In addition, we shall need the assumption

$$\int_0^\infty f(t) e^{-ct} dt < \infty, \tag{2.8}$$

where $c := 4^{-1} \gamma \kappa_{d-1} V_1(B) \mathbb{E} R^{d-1} < \infty$ and $c > 0$ since $V_1(B) > 0$ (recall that B contains a non-degenerate line segment) and $\mathbb{P}(R > 0) > 0$.

Proposition 2.2. Assume that (2.8) is satisfied. If $C \subset \mathbb{R}^+$ is a Borel set and $W_1, W_2 \subset \mathbb{R}^d$ are compact convex sets, then

$$\text{Cov}(\eta_{W_1}(C), \eta_{W_2}(C)) = \int_{\mathbb{R}^d} |W_1 \cap (W_2 - u)|_d [\gamma \tau_1(C, u) + \gamma^2 \tau_2(C, u)] \, du,$$

where

$$\begin{aligned} \tau_1(C, u) := & \int_C \int_{\mathbb{R}^d} \frac{f(d_B(x, B(o, r)))}{h_B(d_B(x, B(o, r)), r)} \frac{f(d_B(u+x, B(o, r)))}{h_B(d_B(u+x, B(o, r)), r)} \\ & \times \bar{F}_B^{(2)}(u; d_B(x, B(o, r)), d_B(u+x, B(o, r))) \, dx \, \mathbb{G}(dr) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} \tau_2(C, u) := & \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_1, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\ & \times q(u; x_1, x_2, r_1, r_2) \, dx_1 \, dx_2 \, \mathbb{G}(dr_1) \, \mathbb{G}(dr_2), \end{aligned} \quad (2.10)$$

for $u \in \mathbb{R}^d$, and

$$\begin{aligned} q(u; x_1, x_2, r_1, r_2) &:= \mathbf{1}\{d_B(x_2, B(u, r_2)) > d_B(x_1, B(o, r_1))\} \mathbf{1}\{d_B(x_1, B(-u, r_1)) > d_B(x_2, B(o, r_2))\} \\ &\times \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) - \bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2))), \end{aligned}$$

for $x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \in \mathbb{R}^+$.

Proof. To abbreviate the notation, we define the function

$$g(t, r) := \mathbf{1}\{r \in C\} f(t) h_B(t, r)^{-1}, \quad (2.11)$$

for $t \in [0, \infty]$ and $r \in \mathbb{R}^+$, with the previous conventions in the cases where $t \in \{0, \infty\}$. Recall also that $h_B(t, r) > 0$ for $t \in (0, \infty)$ and \mathbb{G} -almost all $r \in \mathbb{R}^+$. Using Fubini's theorem and stationarity, we get

$$\begin{aligned} \text{Cov}(\eta_{W_1}(C), \eta_{W_2}(C)) &= \int_{W_1} \int_{W_2} \text{Cov}\left(g(d_B(x_1, Z), r_B(x_1, Z)), g(d_B(x_2, Z), r_B(x_2, Z))\right) \, dx_2 \, dx_1 \\ &= \int_{\mathbb{R}^d} |W_1 \cap (W_2 - u)|_d \text{Cov}\left(g(d_B(o, Z), r_B(o, Z)), g(d_B(u, Z), r_B(u, Z))\right) \, du. \end{aligned}$$

By Lemma 2.1 with $A_1 = A_2 = \mathbb{R}^d$, $x_1 = o$ and $x_2 = u$, we obtain that

$$\text{Cov}\left(g(d_B(o, Z), r_B(o, Z)), g(d_B(u, Z), r_B(u, Z))\right) = J_1(u) + J_{21}(u) - J_{22},$$

where

$$\begin{aligned} J_1(u) := & \gamma \int_0^\infty \int_{\mathbb{R}^d} g(d_B(o, B(x, r)), r) g(d_B(u, B(x, r)), r) \\ & \times \bar{F}_B^{(2)}(u; d_B(o, B(x, r)), d_B(u, B(x, r))) \, dx \, \mathbb{G}(dr), \end{aligned}$$

$$\begin{aligned}
J_{21}(u) &:= \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(o, B(x_1, r_1)), r_1) g(d_B(o, B(x_2, r_2)), r_2) \\
&\quad \times \mathbf{1}\{d_B(-u, B(x_2, r_2)) > d_B(o, B(x_1, r_1))\} \\
&\quad \times \mathbf{1}\{d_B(u, B(x_1, r_1)) > d_B(o, B(x_2, r_2))\} \\
&\quad \times \bar{F}_B^{(2)}(u; d_B(o, B(x_1, r_1)), d_B(o, B(x_2, r_2))) dx_1 dx_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2),
\end{aligned}$$

and

$$\begin{aligned}
J_{22} &:= \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(o, B(x_1, r_1)), r_1) g(d_B(o, B(x_2, r_2)), r_2) \\
&\quad \times \bar{F}_B(d_B(o, B(x_1, r_1))) \bar{F}_B(d_B(o, B(x_2, r_2))) dx_1 dx_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2).
\end{aligned}$$

Using $d_B(u, B(x, r)) = d_B(u - x, B(o, r))$ and the reflection invariance of Lebesgue measure, we deduce that

$$\begin{aligned}
J_1(u) &= \gamma \int_0^\infty \int_{\mathbb{R}^d} g(d_B(x, B(o, r)), r) g(d_B(u + x, B(o, r)), r) \\
&\quad \times \bar{F}_B^{(2)}(u; d_B(x, B(o, r)), d_B(u + x, B(o, r))) dx \mathbb{G}(dr),
\end{aligned}$$

$$\begin{aligned}
J_{21}(u) &= \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, B(o, r_1)), r_1) g(d_B(x_2, B(o, r_2)), r_2) \\
&\quad \times \mathbf{1}\{d_B(x_2, B(u, r_2)) > d_B(x_1, B(o, r_1))\} \\
&\quad \times \mathbf{1}\{d_B(x_1, B(-u, r_1)) > d_B(x_2, B(o, r_2))\} \\
&\quad \times \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) dx_1 dx_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2),
\end{aligned}$$

and

$$\begin{aligned}
J_{22} &= \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, B(o, r_1)), r_1) g(d_B(x_2, B(o, r_2)), r_2) \\
&\quad \times \bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2))) dx_1 dx_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2).
\end{aligned}$$

The assertion now follows by recalling (2.11). The integrability of $\tau_1(C, \cdot)$ and $\tau_2(C, \cdot)$, which is explicitly stated in (2.15), will be shown in the proof of Theorem 2.4 and is implied by the assumption (2.8). \square

Remark 2.3. Recall that $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d . If $B = B^d$ is the unit ball, then $d_{B^d}(u, B(x, r)) = (\|x - u\| - r)^+$,

$$h_{B^d}(t, r) = \sum_{j=0}^{d-1} d\kappa_d \binom{d-1}{j} r^{d-1-j} t^j = d\kappa_d (t + r)^{d-1},$$

and

$$\kappa_{B^d}(u; t_1, t_2, r) = |B(o, t_1 + r) \cap B(u, t_2 + r)|_d.$$

Hence, $\tau_1(C, u)$ and $\tau_2(C, u)$ from Proposition 2.2 may be slightly simplified. In particular, then we have

$$\begin{aligned}
\tau_2(C, u) &= \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(\|x_1\| - r_1)^+}{h_{B^d}(\|x_1\| - r_1)^+, r_1)} \frac{f(\|x_2\| - r_2)^+}{h_{B^d}(\|x_2\| - r_2)^+, r_2)} \\
&\quad \times \left[\mathbf{1} \{ (\|x_2 - u\| - r_2)^+ > (\|x_1\| - r_1)^+ \} \mathbf{1} \{ (\|x_1 + u\| - r_1)^+ > (\|x_2\| - r_2)^+ \} \right. \\
&\quad \times \bar{F}_{B^d}^{(2)}(u; (\|x_1\| - r_1)^+, (\|x_2\| - r_2)^+) - \bar{F}_{B^d}((\|x_1\| - r_1)^+) \bar{F}_{B^d}((\|x_2\| - r_2)^+) \Big] \\
&\quad \times dx_1 dx_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2) \\
&= \int_C \int_C \int_0^\infty \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{f(s_1)}{h_{B^d}(s_1, r_1)} (s_1 + r_1)^{d-1} \frac{f(s_2)}{h_{B^d}(s_2, r_2)} (s_2 + r_2)^{d-1} \\
&\quad \times \left[\mathbf{1} \{ (\|(s_2 + r_2)v_2 - u\| - r_2)^+ > s_1 \} \mathbf{1} \{ (\|(s_1 + r_1)v_1 + u\| - r_1)^+ > s_2 \} \right. \\
&\quad \times \bar{F}_{B^d}^{(2)}(u; s_1, s_2) - \bar{F}_{B^d}(s_1) \bar{F}_{B^d}(s_2) \Big] \mathcal{H}^{d-1}(dv_1) ds_1 \mathcal{H}^{d-1}(dv_2) ds_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2) \\
&= \int_C \int_C \int_0^\infty \int_0^\infty f(s_1) f(s_2) \left[\frac{\mathcal{H}^{d-1}(\partial B(o, s_2 + r_2) \cap B(u, s_1 + r_2)^c)}{\mathcal{H}^{d-1}(\partial B(o, s_2 + r_2))} \right. \\
&\quad \times \frac{\mathcal{H}^{d-1}(\partial B(o, s_1 + r_1) \cap B(-u, s_2 + r_1)^c)}{\mathcal{H}^{d-1}(\partial B(o, s_1 + r_1))} \bar{F}_{B^d}^{(2)}(u; s_1, s_2) - \bar{F}_{B^d}(s_1) \bar{F}_{B^d}(s_2) \Big] \\
&\quad \times ds_1 ds_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2),
\end{aligned}$$

where \mathbb{S}^{d-1} is the unit sphere in \mathbb{R}^d , \mathcal{H}^{d-1} is the $(d-1)$ -dimensional Hausdorff measure, and $\partial B(x, r)$ is the boundary of $B(x, r)$. We used that $f(\|x\| - r)^+$ is non-zero only if $\|x\| > r$. Then $x = (s + r)v$ for $s > 0$ and $v \in \mathbb{S}^{d-1}$.

Next we state a special case of [10, Theorem 2.1 and Remark 3.1] in the form needed in the present context. Let $\tilde{g} : \mathbb{R}^d \rightarrow [0, \infty]$ be measurable, and let $K, B \subset \mathbb{R}^d$ be convex bodies such that $o \in B$ and K, B are in general relative position. Since in our application, we shall only need the case $K = rB^d$, for $r \in \mathbb{R}^+$, the assumption of general relative position will be satisfied for any choice of B . Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{1} \{ 0 < d_B(z, rB^d) < \infty \} \tilde{g}(z) dz \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int t^{d-1-j} \tilde{g}(z + tb) \Theta_{j, d-j}(rB^d; B^*; d(z, b)) dt,
\end{aligned}$$

where $B^* := -B$ and the mixed support measures $\Theta_{j, d-j}(rB^d; B^*; \cdot)$, $j \in \{0, \dots, d-1\}$, are finite Borel measures on \mathbb{R}^{2d} . Using [20, (14.18)] (cf. [18, (4.2.26) and (5.3.8)]) and [12, middle of p. 327], we obtain for the total measures $\Theta_{j, d-j}(rB^d; B^*; \mathbb{R}^{2d}) = r^j d_j^{(d)} \kappa_j V_{d-j}(B)$. In particular, this yields for any measurable function $\tilde{f} : [0, \infty] \rightarrow [0, \infty]$ with $\tilde{f}(0) = \tilde{f}(\infty) = 0$ that

$$\int_{\mathbb{R}^d} \tilde{f}(d_B(z, rB^d)) dz = \int_0^\infty h_B(t, r) \tilde{f}(t) dt. \quad (2.12)$$

We now turn to the asymptotic properties of the ratio-unbiased estimator (1.8). Our setting is similar to [15], where all limit theorems refer to a growing observation window in \mathbb{R}^d . More formally, we

consider a sequence $(W_n)_{n \in \mathbb{N}}$ of compact, convex sets $W_n \subset \mathbb{R}^d$ such that $W_n \subset W_{n+1}$ for all $n \in \mathbb{N}$ and the inradius of W_n tends to ∞ as $n \rightarrow \infty$.

Theorem 2.4. *Assume that (2.8) is fulfilled. Then*

$$\frac{\text{Var } \eta_{W_n}(C)}{|W_n|_d} \xrightarrow{n \rightarrow \infty} \sigma^2(C). \quad (2.13)$$

The asymptotic variance is finite and given by

$$\sigma^2(C) = \gamma \int_{\mathbb{R}^d} \tau_1(C, u) \, du + \gamma^2 \int_{\mathbb{R}^d} \tau_2(C, u) \, du, \quad (2.14)$$

where $\tau_1(C, u)$ and $\tau_2(C, u)$ are defined in (2.9) and (2.10), respectively. Moreover, if $0 < \mathbb{G}(C) < 1$, then $\sigma^2(C) > 0$.

Proof. Suppose that $x \in W_n$ and $u \in \mathbb{R}^d$. If $x + u \notin W_n$, then $d_{B^d}(x, \partial W_n) \leq \|u\|$. Hence, we obtain

$$|W_n|_d - |\{x \in W_n : d_{B^d}(x, \partial W_n) \leq \|u\|\}|_d \leq |W_n \cap (W_n - u)|_d \leq |W_n|_d.$$

Thus, [11, Lemma 10.15 (ii)] implies that

$$\frac{|W_n \cap (W_n - u)|_d}{|W_n|_d} \xrightarrow{n \rightarrow \infty} 1 \quad \text{for any } u \in \mathbb{R}^d.$$

Therefore Lebesgue's dominated convergence theorem and Proposition 2.2 yield (2.13) provided that

$$\int_{\mathbb{R}^d} \tau_1(C, u) \, du < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\tau_2(C, u)| \, du < \infty. \quad (2.15)$$

Using (2.7) we have

$$\begin{aligned} \int_{\mathbb{R}^d} \tau_1(C, u) \, du &\leq \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x, rB^d))}{h_B(d_B(x, rB^d), r)} \frac{f(d_B(y, rB^d))}{h_B(d_B(y, rB^d), r)} \\ &\quad \times \sqrt{\bar{F}_B(d_B(x, rB^d)) \bar{F}_B(d_B(y, rB^d))} \, dx \, dy \, \mathbb{G}(dr) \\ &= \int_C \left(\int_{\mathbb{R}^d} \frac{f(d_B(x, rB^d))}{h_B(d_B(x, rB^d), r)} \sqrt{\bar{F}_B(d_B(x, rB^d))} \, dx \right)^2 \mathbb{G}(dr). \end{aligned}$$

An application of (2.12) shows that

$$\int_{\mathbb{R}^d} \frac{f(d_B(x, rB^d))}{h_B(d_B(x, rB^d), r)} \sqrt{\bar{F}_B(d_B(x, rB^d))} \, dx = \int_0^\infty f(t) \sqrt{\bar{F}_B(t)} \, dt$$

and thus we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \tau_1(C, u) \, du &\leq \int_C \left(\int_0^\infty f(t) \sqrt{\bar{F}_B(t)} \, dt \right)^2 \mathbb{G}(dr) \\ &\leq \mathbb{G}(C) \left(\int_0^\infty f(t) e^{-2ct} \, dt \right)^2 < \infty, \end{aligned}$$

where we use that $\bar{F}_B(t) \leq \exp\{-4ct\}$ and assumption (2.8).

In order to show that

$$\int_{\mathbb{R}^d} |\tau_2(C, u)| \, du < \infty,$$

we first rewrite $q(u; x_1, x_2, r_1, r_2)$ as the difference of two non-negative terms, that is, $q = q_1 - q_2$ with

$$\begin{aligned} q_1(u; x_1, x_2, r_1, r_2) &:= \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) \\ &\quad - \bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2))), \end{aligned}$$

which is non-negative by (2.6), and

$$\begin{aligned} q_2(u; x_1, x_2, r_1, r_2) &:= \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) \\ &\quad \times \left(1 - \mathbf{1}\{d_B(x_2, B(u, r_2)) > d_B(x_1, B(o, r_1))\} \mathbf{1}\{d_B(x_1, B(-u, r_1)) > d_B(x_2, B(o, r_2))\}\right), \end{aligned}$$

for $u, x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \in \mathbb{R}^+$. Using (2.4), (2.7) and the inequality $1 - e^{-a} \leq a$, for $a \geq 0$, we get

$$\begin{aligned} q_1(u; x_1, x_2, r_1, r_2) &\leq \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) \\ &\quad \times \left(1 - \exp\{-\gamma \mathbb{E} \kappa_B(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2)), R)\}\right) \\ &\leq \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2)))} \\ &\quad \times \gamma \mathbb{E} \kappa_B(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2)), R). \end{aligned}$$

Moreover, the inequality $1 - (1 - a)(1 - b) \leq a + b$, for $a, b \geq 0$, and again (2.7) imply that

$$\begin{aligned} q_2(u; x_1, x_2, r_1, r_2) &\leq \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2)))} \\ &\quad \times \left(\mathbf{1}\{d_B(x_2, B(u, r_2)) \leq d_B(x_1, B(o, r_1))\} \right. \\ &\quad \left. + \mathbf{1}\{d_B(x_1, B(-u, r_1)) \leq d_B(x_2, B(o, r_2))\}\right). \end{aligned}$$

Combining these bounds, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^d} |\tau_2(C, u)| \, du &\leq \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_1, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\ &\quad \times \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2)))} \\ &\quad \times \left[\gamma \mathbb{E} \kappa_B(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2)), R) \right. \\ &\quad \left. + \mathbf{1}\{d_B(x_2, B(u, r_2)) \leq d_B(x_1, B(o, r_1))\} \right. \\ &\quad \left. + \mathbf{1}\{d_B(x_1, B(-u, r_1)) \leq d_B(x_2, B(o, r_2))\} \right] \\ &\quad \times \, dx_1 \, dx_2 \, du \, \mathbb{G}(dr_1) \, \mathbb{G}(dr_2). \end{aligned}$$

The preceding expression splits naturally into three summands which will be bounded from above separately. For the first bound, we observe that by Fubini's theorem

$$\mathbb{E} \int_{\mathbb{R}^d} \kappa_B(u; s_1, s_2, R) \, du = \mathbb{E} |B_{s_1, R}|_d |B_{s_2, R}|_d.$$

Then we apply (2.12) to get

$$\begin{aligned}
& \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_1, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\
& \quad \times \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2)))} \\
& \quad \times \gamma \mathbb{E} \kappa_B(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2)), R) \\
& \quad \times dx_1 dx_2 du \mathbb{G}(dr_1) \mathbb{G}(dr_2) \\
& = \gamma \mathbb{G}(C)^2 \int_0^\infty \int_0^\infty f(t_1) \sqrt{\bar{F}_B(t_1)} f(t_2) \sqrt{\bar{F}_B(t_2)} \mathbb{E}|B_{t_1, R}|_d |B_{t_2, R}|_d dt_1 dt_2.
\end{aligned}$$

Choose $c_B > 0$ such that $B \subset c_B B^d$. Then $|B_{t, R}|_d \leq \kappa_d(c_B t + R)^d$ and hence the Cauchy-Schwarz inequality, the convexity of $s \mapsto s^p$, $p \geq 1$, and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $a, b \geq 0$, yield that

$$\begin{aligned}
\mathbb{E}|B_{t_1, R}|_d |B_{t_2, R}|_d & \leq c_1 \sqrt{\mathbb{E}(c_B t_1 + R)^{2d}} \sqrt{\mathbb{E}(c_B t_2 + R)^{2d}} \\
& \leq c_2 \left(c_B^d t_1^d + \sqrt{\mathbb{E} R^{2d}} \right) \left(c_B^d t_2^d + \sqrt{\mathbb{E} R^{2d}} \right) \\
& \leq c_3 (1 + t_1^d) (1 + t_2^d),
\end{aligned}$$

where c_1, c_2, c_3 denote finite constants independent of the expectation or t_1, t_2 . From this and (2.8) it follows again that the first summand is finite.

Since $d_B(x_2, B(u, r_2)) \leq t_1$ if and only if $u \in x_2 + B_{t_1, r_2}$, applying Fubini's theorem and (2.12) (twice) we obtain for the second summand that

$$\begin{aligned}
& \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_1, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\
& \quad \times \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2)))} \\
& \quad \times \mathbf{1}\{d_B(x_2, B(u, r_2)) \leq d_B(x_1, B(o, r_1))\} \\
& \quad \times dx_1 dx_2 du \mathbb{G}(dr_1) \mathbb{G}(dr_2) \\
& = \int_C \int_C \int_0^\infty \int_0^\infty f(t_1) \sqrt{\bar{F}_B(t_1)} f(t_2) \sqrt{\bar{F}_B(t_2)} |B_{t_1, r_2}|_d dt_1 dt_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2),
\end{aligned}$$

which is finite by the same reasoning as above.

The third summand can be treated in exactly the same way.

To prove positivity of the asymptotic variance we use the fact that the variance of any square-integrable function $H(\Psi)$ of the Poisson process $\Psi := \{(\xi_n, R_n) : n \geq 1\}$ satisfies the inequality

$$\text{Var } H(\Psi) \geq \gamma \int_0^\infty \int_{\mathbb{R}^d} (\mathbb{E}[H(\Psi \cup \{(y, r)\})] - H(\Psi))^2 dy \mathbb{G}(dr);$$

see, e.g., [13, Theorem 4.2]. In our case this means that

$$\text{Var } \eta_W(C) \geq \gamma \int_0^\infty \int_{\mathbb{R}^d} \tilde{h}(y, r)^2 dy \mathbb{G}(dr), \tag{2.16}$$

where

$$\begin{aligned}
\tilde{h}(y, r) &:= \mathbb{E} \int_W [g(d_B(x, Z \cup B(y, r)), r_B(x, Z \cup B(y, r))) - g(d_B(x, Z), r_B(x, Z))] dx \\
&= \mathbb{E} \int_W \mathbf{1}\{d_B(x, B(y, r)) < d_B(x, Z)\} [g(d_B(x, B(y, r)), r) - g(d_B(x, Z), r_B(x, Z))] dx \\
&= \mathbb{E} \int_W \mathbf{1}\{d_B(o, B(y - x, r)) < d_B(o, Z)\} \\
&\quad \times [g(d_B(o, B(y - x, r)), r) - g(d_B(o, Z), r_B(o, Z))] dx.
\end{aligned}$$

Here the last identity follows from the stationarity of Z and g is as defined in (2.11). By (1.4),

$$\begin{aligned}
\tilde{h}(y, r) &= \gamma \int_W \int_0^\infty \int_0^\infty \mathbf{1}\{d_B(o, B(y - x, r)) < t\} \\
&\quad \times [g(d_B(o, B(y - x, r)), r) - g(t, s)] h_B(t, s) \bar{F}_B(t) dt \mathbb{G}(ds) dx.
\end{aligned}$$

Assume now that $0 < \mathbb{G}(C) < 1$ and let $C' := \mathbb{R}^+ \setminus C$. Recalling the definition (2.11) of g , we obtain from (2.16) that

$$\text{Var } \eta_W(C) \geq \gamma \int_0^\infty \int_{\mathbb{R}^d} h^*(y, r)^2 \mathbf{1}\{r \in C', y \in W\} dy \mathbb{G}(dr),$$

where

$$\begin{aligned}
h^*(y, r) &:= \gamma \int_W \int_0^\infty \int_0^\infty \mathbf{1}\{d_B(o, B(y - x, r)) < t\} g(t, s) h_B(t, s) \bar{F}_B(t) dt \mathbb{G}(ds) dx \\
&= \gamma \mathbb{G}(C) \int_W \int_0^\infty \mathbf{1}\{d_B(o, B(y - x, r)) < t\} f(t) \bar{F}_B(t) dt dx.
\end{aligned}$$

Applying Jensen's inequality with the normalization of $\mathbf{1}\{r \in C', y \in W\} dy \mathbb{G}(dr)$, we get

$$\text{Var } \eta_W(C) \geq \frac{\gamma}{\mathbb{G}(C')|W|_d} \left(\int_{C'} \int_W h^*(y, r) dy \mathbb{G}(dr) \right)^2.$$

Letting $a := \gamma^3 \mathbb{G}(C)^2 / \mathbb{G}(C') > 0$ we obtain that

$$\begin{aligned}
&\frac{\text{Var } \eta_W(C)}{|W|_d} \\
&\geq \frac{a}{|W|_d^2} \left(\int_{C'} \int_W \int_W \int_0^\infty \mathbf{1}\{d_B(o, B(y - x, r)) < t\} f(t) \bar{F}_B(t) dt dx dy \mathbb{G}(dr) \right)^2 \\
&= \frac{a}{|W|_d^2} \left(\int_{C'} \int_{\mathbb{R}^d} \int_0^\infty |W \cap (W - y)|_d \mathbf{1}\{d_B(o, B(y, r)) < t\} f(t) \bar{F}_B(t) dt dy \mathbb{G}(dr) \right)^2.
\end{aligned}$$

Hence it is sufficient to show that

$$\int_{C'} \int_0^\infty \left(\int_{\mathbb{R}^d} \mathbf{1}\{d_B(o, B(y, r)) < t\} dy \right) f(t) \bar{F}_B(t) dt \mathbb{G}(dr) > 0.$$

This is true, since the inner integral is positive for all $r, t > 0$ and since both $\int_0^\infty f(t) \bar{F}_B(t) dt$ and $\mathbb{G}(C')$ are positive. \square

Remark 2.5. The assumption (2.8) is slightly stronger than (1.6).

Remark 2.6. Let $\widehat{\mathbb{G}}_n(C)$ be given by (1.8) with $W = W_n$. Theorem 2.4 implies that $\widehat{\mathbb{G}}_n(C)$ is asymptotically weakly consistent. Indeed, (1.7) and

$$\frac{\text{Var } \eta_{W_n}(C)}{|W_n|_d^2} \xrightarrow[n \rightarrow \infty]{} 0$$

ensure that $\eta_{W_n}(C)/|W_n|_d$ converges to $\gamma \beta \mathbb{G}(C)$ in probability as $n \rightarrow \infty$. Especially,

$$\frac{\eta_{W_n}(\mathbb{R}^+)}{|W_n|_d} \xrightarrow[n \rightarrow \infty]{} \gamma \beta \quad \text{in probability.} \quad (2.17)$$

Hence, by the continuous mapping theorem, $\eta_{W_n}(C)/\eta_W(\mathbb{R}^+)$ converges to $\mathbb{G}(C)$ in probability as $n \rightarrow \infty$. This is in accordance with the following proposition which even shows that $\widehat{\mathbb{G}}_n(C)$ is asymptotically strongly consistent.

Proposition 2.7. *For any Borel set $C \subset \mathbb{R}^+$, we have $\widehat{\mathbb{G}}_n(C) \xrightarrow[n \rightarrow \infty]{} \mathbb{G}(C)$ \mathbb{P} -a.s.*

Proof. The mapping $W \mapsto \eta_W(C)$ defined by (1.5) is a random measure on \mathbb{R}^d depending on the Boolean model Z in a translation-invariant way. As the Boolean model is ergodic (see [20, Theorem 9.3.5]) we can apply the spatial ergodic theorem (see [11, Corollary 10.19]) to conclude that

$$\lim_{n \rightarrow \infty} |W_n|_d^{-1} \eta_{W_n}(C) = \mathbb{E} \eta_{[0,1]^d}(C) = \gamma \beta \mathbb{G}(C) \quad \mathbb{P}\text{-a.s.}$$

Applying this to the numerator as well as to the denominator in (1.8), we obtain the desired result. \square

3 Asymptotic normality

In this section we study the asymptotic normality of the ratio-unbiased estimator (1.8) for the radius distribution \mathbb{G} of our stationary Boolean model Z with spherical grains. The proof will be based on approximation by m -dependent random fields. This idea comes from [7], where the same technique was used to prove the central limit theorem for random measures which are associated with the Boolean model in an additive way. In contrast to [7], the contribution of an individual grain to the random measure $A \mapsto \eta_A(C)$ is not determined by the grain alone, but does depend on a random number of other grains in a non-trivial manner. Therefore the results of [7] do not apply in our setting.

We consider, for $n \in \mathbb{N}$ and a Borel set $C \subset \mathbb{R}^+$, the estimator

$$\widehat{\mathbb{G}}_n(C) = \frac{\eta_{W_n}(C)}{\eta_{W_n}(\mathbb{R}^d)},$$

where $W_n := [-n, n]^d$ and η_{W_n} is given in (1.5). First we concentrate on the asymptotic normality of the numerator $\eta_{W_n}(C)$. In addition to (1.6), we shall need the integrability condition

$$\int_0^\infty (1+t^d) f(t) dt < \infty, \quad (3.1)$$

which is more restrictive than (2.8).

Theorem 3.1. *Assume that (1.6) and (3.1) are fulfilled. Then, for any Borel set $C \subset \mathbb{R}^+$,*

$$\sqrt{|W_n|_d} \left(\frac{\eta_{W_n}(C)}{|W_n|_d} - \gamma \beta \mathbb{G}(C) \right) \xrightarrow[n \rightarrow \infty]{d} N(0, \sigma^2(C)),$$

where $\sigma^2(C)$ is given by (2.14).

Proof. We fix a Borel set $C \subset \mathbb{R}^+$ and skip the dependence on C in the notation. Let $E_z := [0, 1]^d + z$ for $z \in I_n := \{-n, \dots, n-1\}^d$. Then

$$\eta_{W_n} = \sum_{z \in I_n} \eta_{E_z} = \sum_{z \in I_n} \int_{E_z} g(d_B(x, Z), r_B(x, Z)) dx,$$

where g is given by (2.11). For some fixed integer m , we put $F_z := E_z \oplus [-m, m]^d$. We decompose η_{E_z} into two random variables

$$\eta_z^{(m)} := \int_{E_z} g(d_B(x, Z(F_z)), r_B(x, Z(F_z))) dx$$

and $\tilde{\eta}_z^{(m)} := \eta_{E_z} - \eta_z^{(m)}$. Let $\eta_{W_n}^{(m)} := \sum_{z \in I_n} \eta_z^{(m)}$ and $\tilde{\eta}_{W_n}^{(m)} := \sum_{z \in I_n} \tilde{\eta}_z^{(m)}$ so that $\eta_{W_n} = \eta_{W_n}^{(m)} + \tilde{\eta}_{W_n}^{(m)}$. It is easily seen that $\{\eta_u^{(m)} : u \in U\}$ and $\{\eta_v^{(m)} : v \in V\}$ are independent whenever $U, V \subset \mathbb{Z}^d$ are such that $\|u - v\|_\infty > 2m$ for each $u \in U$ and $v \in V$. Thus, the random variables $\eta_z^{(m)}$, for $z \in \mathbb{Z}^d$, constitute a stationary $(2m)$ -dependent random field (cf. [6, Section 4.3.1]). The variance of $\eta_{W_n}^{(m)}$ is

$$\text{Var } \eta_{W_n}^{(m)} = \text{Var} \sum_{z \in I_n} \eta_z^{(m)} = \sum_{z_1 \in I_n} \sum_{z_2 \in I_n} \text{Cov}(\eta_{z_1}^{(m)}, \eta_{z_2}^{(m)}) = \sum_{z \in I_n - I_n} N_n(z) \text{Cov}(\eta_o^{(m)}, \eta_z^{(m)}),$$

where $N_n(z)$ is the cardinality of $\{(z_1, z_2) \in I_n \times I_n : z_2 - z_1 = z\}$, which may be bounded by $|W_n|_d = (2n)^d$ and $\lim_{n \rightarrow \infty} N_n(z)/|W_n|_d = 1$ for any $z \in \mathbb{Z}^d$. We define

$$(\sigma_n^{(m)})^2 := \frac{\text{Var } \eta_{W_n}^{(m)}}{|W_n|_d}.$$

Since $\text{Cov}(\eta_o^{(m)}, \eta_z^{(m)}) = 0$ for $\|z\| > 2m$, the limit of $(\sigma_n^{(m)})^2$ as $n \rightarrow \infty$ exists and satisfies

$$(\sigma^{(m)})^2 := \lim_{n \rightarrow \infty} (\sigma_n^{(m)})^2 = \sum_{z \in \{-2m, \dots, 2m\}^d} \text{Cov}(\eta_o^{(m)}, \eta_z^{(m)}). \quad (3.2)$$

Next we show that $\mathbb{E}(\eta_o^{(m)})^2 < \infty$. We put $A := [-m, m+1]^d$, hence

$$\eta_o^{(m)} = \int_{E_o} g(d_B(x, Z(A)), r_B(x, Z(A))) dx.$$

Proceeding as in the proof of Proposition 2.2 and bounding $\bar{F}_B^{A,A}(\cdot)$ as well as $(1 - \mathbf{1}\{\cdot\})\mathbf{1}\{\cdot\}$ by 1, we get

$$\begin{aligned} \mathbb{E}(\eta_o^{(m)})^2 &\leq \gamma \int_{E_o} \int_{E_o} \int_0^\infty \int_A g(d_B(y, B(x_1, r)), r) g(d_B(y, B(x_2, r)), r) dy \mathbb{G}(dr) dx_1 dx_2 \\ &\quad + \gamma^2 \int_{E_o} \int_{E_o} \int_0^\infty \int_0^\infty \int_A \int_A g(d_B(y_1, B(x_1, r_1)), r_1) \\ &\quad \times g(d_B(y_2, B(x_2, r_2)), r_2) dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2) dx_1 dx_2. \end{aligned}$$

The right-hand side increases if A is replaced by \mathbb{R}^d . Arguing then as in the proof of Theorem 2.4, we obtain

$$\mathbb{E}(\eta_o^{(m)})^2 \leq \gamma |E_o|_d \left(\int_0^\infty f(t) dt \right)^2 \mathbb{G}(C) + \left(\gamma |E_o|_d \int_0^\infty f(t) dt \mathbb{G}(C) \right)^2 < \infty.$$

Therefore, the central limit theorem for stationary m -dependent random fields (see, e.g., [17]) yields that

$$\frac{1}{\sqrt{|W_n|_d}} \sum_{z \in I_n} \left(\eta_z^{(m)} - \mathbb{E} \eta_z^{(m)} \right) \xrightarrow[n \rightarrow \infty]{d} N(0, (\sigma^{(m)})^2).$$

In view of [2, Theorem 3.2], it remains to verify that

$$\lim_{m \rightarrow \infty} \sigma^{(m)} = \sigma(C) \quad (3.3)$$

and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{\sqrt{|W_n|_d}} \left| \sum_{z \in I_n} (\tilde{\eta}_z^{(m)} - \mathbb{E} \tilde{\eta}_z^{(m)}) \right| \geq \varepsilon \right) = 0 \quad \text{for any } \varepsilon > 0. \quad (3.4)$$

Define

$$\sigma_n^2 := \frac{\text{Var } \eta_{W_n}}{|W_n|_d}.$$

Then

$$|\sigma(C) - \sigma^{(m)}| \leq |\sigma(C) - \sigma_n| + |\sigma_n - \sigma_n^{(m)}| + |\sigma_n^{(m)} - \sigma^{(m)}|.$$

The first term goes to zero as $n \rightarrow \infty$ by Theorem 2.4, the last term goes to zero as $n \rightarrow \infty$ as well, for any $m \in \mathbb{N}$, by (3.2). By Minkowski's inequality, the middle term can be bounded as

$$|\sigma_n - \sigma_n^{(m)}| \leq \frac{1}{\sqrt{|W_n|_d}} \sqrt{\text{Var } \tilde{\eta}_{W_n}^{(m)}}.$$

Therefore, (3.3) follows if we can show that

$$\sup_{n \in \mathbb{N}} \frac{1}{|W_n|_d} \text{Var } \tilde{\eta}_{W_n}^{(m)} \xrightarrow{m \rightarrow \infty} 0. \quad (3.5)$$

By Chebyshev's inequality, (3.5) also implies (3.4). The variance in (3.5) satisfies

$$\frac{1}{|W_n|_d} \text{Var} \sum_{z \in I_n} \tilde{\eta}_z^{(m)} = \frac{1}{|W_n|_d} \sum_{z \in I_n - I_n} N_n(z) \text{Cov}(\tilde{\eta}_0^{(m)}, \tilde{\eta}_z^{(m)}) \leq \sum_{z \in \mathbb{Z}^d} |\text{Cov}(\tilde{\eta}_0^{(m)}, \tilde{\eta}_z^{(m)})|.$$

Therefore, the proof will be finished when we show that

$$\sum_{z \in \mathbb{Z}^d} |\text{Cov}(\tilde{\eta}_0^{(m)}, \tilde{\eta}_z^{(m)})| \xrightarrow{m \rightarrow \infty} 0.$$

Consider a fixed $z \in \mathbb{Z}^d$. Then the covariance can be written as

$$\begin{aligned} \text{Cov}(\tilde{\eta}_0^{(m)}, \tilde{\eta}_z^{(m)}) &= \text{Cov}(\eta_{E_0}, \eta_{E_z}) - \text{Cov}(\eta_0^{(m)}, \eta_{E_z}) - \text{Cov}(\eta_{E_0}, \eta_z^{(m)}) + \text{Cov}(\eta_0^{(m)}, \eta_z^{(m)}) \\ &= \int_{E_0} \int_{E_z} [c_{\mathbb{R}^d, \mathbb{R}^d}(x_1, x_2) - c_{F_0, \mathbb{R}^d}(x_1, x_2) - c_{\mathbb{R}^d, F_z}(x_1, x_2) + c_{F_0, F_z}(x_1, x_2)] dx_2 dx_1, \end{aligned}$$

where, for Borel sets $A_1, A_2 \subset \mathbb{R}^d$ and $x_1, x_2 \in \mathbb{R}^d$,

$$c_{A_1, A_2}(x_1, x_2) := \text{Cov} \left(g(d_B(x_1, Z(A_1))), r_B(x_1, Z(A_1))), g(d_B(x_2, Z(A_2))), r_B(x_2, Z(A_2))) \right)$$

is expressed in Lemma 2.1 as

$$\begin{aligned} c_{A_1, A_2}(x_1, x_2) &= \gamma \int_0^\infty \int_{\mathbb{R}^d} g(d_B(x_1, B(y, r)), r) g(d_B(x_2, B(y, r)), r) I_1(A_1, A_2) dy \mathbb{G}(dr) \\ &\quad + \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, B(y_1, r_1)), r_1) g(d_B(x_2, B(y_2, r_2)), r_2) \\ &\quad \times I_2(A_1, A_2) dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2). \end{aligned}$$

Here we skip the arguments x_1, x_2, y, r , respectively $x_1, x_2, y_1, y_2, r_1, r_2$, of the functions $I_1(A_1, A_2)$ and $I_2(A_1, A_2)$, which were defined before Lemma 2.1. We shall treat both parts of $c_{A_1, A_2}(x_1, x_2)$ separately. Our aim is to prove that

$$\begin{aligned} S_1 &:= \sum_{z \in \mathbb{Z}^d} \int_{E_o} \int_{E_z} \int_0^\infty \int_{\mathbb{R}^d} g(d_B(x_1, B(y, r)), r) g(d_B(x_2, B(y, r)), r) \\ &\quad \times \left| I_1(\mathbb{R}^d, \mathbb{R}^d) - I_1(F_o, \mathbb{R}^d) - I_1(\mathbb{R}^d, F_z) + I_1(F_o, F_z) \right| dy \mathbb{G}(dr) dx_2 dx_1 \end{aligned}$$

and

$$\begin{aligned} S_2 &:= \sum_{z \in \mathbb{Z}^d} \int_{E_o} \int_{E_z} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, B(y_1, r_1)), r_1) g(d_B(x_2, B(y_2, r_2)), r_2) \\ &\quad \times \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2) dx_2 dx_1 \end{aligned}$$

tend to zero as $m \rightarrow \infty$. Observe that S_1, S_2 depend on m via the dependence of F_o, F_z on m .

First, we consider S_1 . We rewrite

$$\begin{aligned} &I_1(\mathbb{R}^d, \mathbb{R}^d) - I_1(F_o, \mathbb{R}^d) - I_1(\mathbb{R}^d, F_z) + I_1(F_o, F_z) \\ &= \mathbf{1}\{y \in F_o^c \cap F_z^c\} \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d}(x_1, x_2; t_1, t_2) + \mathbf{1}\{y \in F_o \cap F_z^c\} \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right)(x_1, x_2; t_1, t_2) \\ &\quad + \mathbf{1}\{y \in F_o^c \cap F_z\} \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} \right)(x_1, x_2; t_1, t_2) \\ &\quad + \mathbf{1}\{y \in F_o \cap F_z\} \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right)(x_1, x_2; t_1, t_2) \end{aligned}$$

with $t_1 = d_B(x_1, B(y, r))$ and $t_2 = d_B(x_2, B(y, r))$. For notational simplicity, write

$$\nu_1(x_1, t_1) := \gamma \mathbb{E}[(x_1 + B_{t_1, R}) \cap F_o^c]_d, \quad \nu_2(x_2, t_2) := \gamma \mathbb{E}[(x_2 + B_{t_2, R}) \cap F_z^c]_d$$

for $x_1, x_2 \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$. We suppress the dependence on z in $\nu_2(x_2, t_2)$. From (2.3) and the inequality $1 - e^{-a} \leq a$, for $a \geq 0$, we obtain for $x_1, x_2 \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$ that

$$\begin{aligned} &\left| \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right)(x_1, x_2; t_1, t_2) \right| = \left(\bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} \right)(x_1, x_2; t_1, t_2) \\ &= \bar{F}_B^{F_o, \mathbb{R}^d}(x_1, x_2; t_1, t_2) \left(1 - \exp \{ -\gamma \mathbb{E}[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R})^c \cap F_o^c]_d \} \right) \\ &\leq \nu_1(x_1, t_1). \end{aligned} \tag{3.6}$$

Analogously,

$$\left| \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} \right)(x_1, x_2; t_1, t_2) \right| \leq \nu_2(x_2, t_2). \tag{3.7}$$

Furthermore,

$$\begin{aligned}
& \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) = \bar{F}_B^{F_o, F_z} (x_1, x_2; t_1, t_2) \\
& \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left| (x_1 + B_{t_1, R}) \cap F_o^c \cap ((x_2 + B_{t_2, R})^c \cup F_z^c) \right|_d \right\} \right) \\
& \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left| (x_2 + B_{t_2, R}) \cap F_z^c \cap ((x_1 + B_{t_1, R})^c \cup F_o^c) \right|_d \right\} \right) \\
& + \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} (x_1, x_2; t_1, t_2) \left(1 - \exp \left\{ -\gamma \mathbb{E} \left| (x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o^c \cap F_z^c \right|_d \right\} \right) \quad (3.8)
\end{aligned}$$

gives

$$\begin{aligned}
& \left| \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \right| \\
& \leq \nu_1(x_1, t_1) \nu_2(x_2, t_2) + \sqrt{\nu_1(x_1, t_1) \nu_2(x_1, t_2)} \quad (3.9)
\end{aligned}$$

because by the Cauchy-Schwarz inequality

$$\mathbb{E} |X_1 \cap X_2|_d \leq \sqrt{\mathbb{E} |X_1|_d} \sqrt{\mathbb{E} |X_2|_d}$$

for any random sets X_1 and X_2 . Combining (3.6), (3.7) and (3.9), we obtain

$$\begin{aligned}
& \left| I_1(\mathbb{R}^d, \mathbb{R}^d) - I_1(F_o, \mathbb{R}^d) - I_1(\mathbb{R}^d, F_z) + I_1(F_o, F_z) \right| \\
& \leq \mathbf{1}\{y \in F_o^c \cap F_z^c\} + \mathbf{1}\{y \in F_o \cap F_z^c\} \nu_1(x_1, t_1) + \mathbf{1}\{y \in F_o^c \cap F_z\} \nu_2(x_2, t_2) \\
& + \mathbf{1}\{y \in F_o \cap F_z\} \left[\nu_1(x_1, t_1) \nu_2(x_2, t_2) + \sqrt{\nu_1(x_1, t_1) \nu_2(x_2, t_2)} \right], \quad (3.10)
\end{aligned}$$

where $t_1 = d_B(x_1, B(y, r))$ and $t_2 = d_B(x_2, B(y, r))$. If $x_1 \in E_o$ and $x_2 \in E_z$, then $\nu_1(x_1, t_1)$ is bounded by $\nu(t_1)$ and $\nu_2(x_2, t_2)$ is bounded by $\nu(t_2)$, where

$$\nu(t) := \gamma \mathbb{E} \left| (E_o \oplus B_{t, R}) \cap F_o^c \right|_d, \quad t \in \mathbb{R}^+.$$

Let $c_B > 0$ be such that $B \subset c_B B^d$. Then

$$\nu(t) \leq \mathbb{E} |B_{t, R+\sqrt{d}}|_d \leq \kappa_d \mathbb{E} (c_B t + R + \sqrt{d})^d \leq c_1 (1 + t^d), \quad (3.11)$$

where c_1 is a finite constant that does not depend on t . If $x_1 \in E_o$ and $y \in F_o^c$, then $\|x_1 - y\| \geq m$ and $d_B(x_1, B(y, r)) \geq (m - r)^+/c_B$. Similarly, if $x_2 \in E_z$ and $y \in F_z^c$, then $d_B(x_2, B(y, r)) \geq (m - r)^+/c_B$. Let

$$\psi(t, r) := \mathbf{1} \left\{ t \geq \frac{(m - r)^+}{c_B} \right\}, \quad t, r \in \mathbb{R}^+.$$

Then, by (3.10) and the substitutions $x_1 - y \rightarrow x_1$ and $x_2 - y \rightarrow x_2$, we get

$$\begin{aligned}
S_1 & \leq \sum_{z \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} g(d_B(x_1, rB^d), r) g(d_B(x_2, rB^d), r) \mathbf{1}\{x_1 + y \in E_o, x_2 + y \in E_z\} \\
& \times \left[\psi(d_B(x_1, rB^d), r) \psi(d_B(x_2, rB^d), r) + \psi(d_B(x_2, rB^d), r) \nu(d_B(x_1, rB^d)) \right. \\
& + \psi(d_B(x_1, rB^d), r) \nu(d_B(x_2, rB^d)) + \nu(d_B(x_1, rB^d)) \nu(d_B(x_2, rB^d)) \\
& \left. + \sqrt{\nu(d_B(x_1, rB^d)) \nu(d_B(x_2, rB^d))} \right] dy \mathbb{G}(dr) dx_2 dx_1,
\end{aligned}$$

and thus

$$S_1 \leq \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, rB^d), r) g(d_B(x_2, rB^d), r) \\ \times v(d_B(x_1, rB^d), d_B(x_2, rB^d), r) dx_2 dx_1 \mathbb{G}(dr),$$

where

$$v(t_1, t_2, r) := (\psi(t_1, r) + \nu(t_1))(\psi(t_2, r) + \nu(t_2)) + \sqrt{\nu(t_1)\nu(t_2)}, \quad t_1, t_2, r \in \mathbb{R}^+.$$

Now Fubini's theorem, two applications of (2.12) and definition (2.11) of g yield

$$S_1 \leq \int_C \int_0^\infty \int_0^\infty f(t_1) f(t_2) v(t_1, t_2, r) dt_2 dt_1 \mathbb{G}(dr).$$

Our moment assumption (1.2) ensures that $\mathbb{E}|E_o \oplus B_{t,R}|_d < \infty$ and therefore $\nu(t) \rightarrow 0$ as $m \rightarrow \infty$ by Lebesgue's dominated convergence theorem. By (3.11) and (3.1), another application of Lebesgue's dominated convergence theorem shows that $S_1 \rightarrow 0$ as $m \rightarrow \infty$.

Next, we proceed with S_2 . From (2.4) and the inequality $1 - e^{-a} \leq a$, for $a \geq 0$, we obtain for $x_1, x_2 \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$ that

$$\begin{aligned} & \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d}(x_1, x_2; t_1, t_2) - \bar{F}_B^{\mathbb{R}^d}(x_1; t_1) \bar{F}_B^{\mathbb{R}^d}(x_2; t_2) \\ &= \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d}(x_1, x_2; t_1, t_2) \left(1 - \exp\{-\gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R)\}\right) \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R), \end{aligned} \tag{3.12}$$

where κ_B is defined in (2.5), and

$$\begin{aligned} & \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right)(x_1, x_2; t_1, t_2) - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right)(x_1; t_1) \bar{F}_B^{\mathbb{R}^d}(x_2; t_2) \\ &= -\bar{F}_B^{F_o, \mathbb{R}^d}(x_1, x_2; t_1, t_2) \left(1 - \exp\{-\gamma \mathbb{E}|(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o|_d\}\right) \\ &\quad + \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d}(x_1, x_2; t_1, t_2) \left(1 - \exp\{-\gamma \mathbb{E}|(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R})|_d\}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \left| \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right)(x_1, x_2; t_1, t_2) - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right)(x_1; t_1) \bar{F}_B^{\mathbb{R}^d}(x_2; t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R). \end{aligned} \tag{3.13}$$

Analogously,

$$\begin{aligned} & \left| \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} \right)(x_1, x_2; t_1, t_2) - \bar{F}_B^{\mathbb{R}^d}(x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right)(x_2; t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R). \end{aligned} \tag{3.14}$$

Finally, using (3.8) we get

$$\begin{aligned}
& \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \\
& \quad - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \\
& = \bar{F}_B^{F_o, F_z} (x_1, x_2; t_1, t_2) \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap F_o^c \cap ((x_2 + B_{t_2, R})^c \cup F_z^c) \middle|_d \right] \right\} \right) \\
& \quad \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_2 + B_{t_2, R}) \cap F_z^c \cap ((x_1 + B_{t_1, R})^c \cup F_o^c) \middle|_d \right] \right\} \right) \\
& \quad + \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} (x_1, x_2; t_1, t_2) \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o^c \cap F_z^c \middle|_d \right] \right\} \right) \\
& \quad - \bar{F}_B^{F_o} (x_1; t_1) \bar{F}_B^{F_z} (x_2; t_2) (1 - \exp \{ -\nu_1(x_1, t_1) \}) (1 - \exp \{ -\nu_2(x_2, t_2) \}) \\
& = \bar{F}_B^{F_o, F_z} (x_1, x_2; t_1, t_2) \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap F_o^c \cap ((x_2 + B_{t_2, R})^c \cup F_z^c) \middle|_d \right] \right\} \right) \\
& \quad \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_2 + B_{t_2, R}) \cap F_z^c \cap ((x_1 + B_{t_1, R})^c \cup F_o^c) \middle|_d \right] \right\} \right) \\
& \quad \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o \cap F_z \middle|_d \right] \right\} \right) \\
& \quad + \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} (x_1, x_2; t_1, t_2) \left(1 - \exp \{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o^c \cap F_z^c \middle|_d \right] \right) \\
& \quad - \bar{F}_B^{F_o} (x_1; t_1) \bar{F}_B^{F_z} (x_2; t_2) \left[\exp \left\{ -\gamma \mathbb{E} \left[(x_2 + B_{t_2, R}) \cap F_z^c \cap ((x_1 + B_{t_1, R})^c \cup F_o^c) \middle|_d \right] \right\} \right. \\
& \quad \times \left(1 - \exp \{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o \cap F_z \middle|_d \right] \} \right) (1 - \exp \{ -\nu_1(x_1, t_1) \}) \\
& \quad + \exp \left\{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap F_o^c \cap ((x_2 + B_{t_2, R})^c \cup F_z^c) \middle|_d \right] \right\} \\
& \quad \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o^c \cap F_z \middle|_d \right] \right\} \right) \\
& \quad \left. \times \left(1 - \exp \left\{ -\gamma \mathbb{E} \left[(x_2 + B_{t_2, R}) \cap F_z^c \cap ((x_1 + B_{t_1, R})^c \cup F_o^c) \middle|_d \right] \right\} \right) \right],
\end{aligned}$$

which leads to

$$\begin{aligned}
& \left| \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \right. \\
& \quad \left. - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\
& \leq \nu_1(x_1, t_1) \nu_2(x_2, t_2) \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) \\
& \quad + \gamma \mathbb{E} \left[(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o^c \cap F_z^c \middle|_d \right] \\
& \quad + \nu_1(x_1, t_1) \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) + \nu_2(x_2, t_2) \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R). \tag{3.15}
\end{aligned}$$

In the following, we use (3.12) – (3.15) with $t_1 = d_B(x_1, B(y_1, r_1))$, $t_2 = d_B(x_2, B(y_2, r_2))$. Moreover, we define

$$\begin{aligned}
\chi_1 &:= \mathbf{1} \{ d_B(x_1, B(y_2, r_2)) \leq t_1 \} = \mathbf{1} \{ x_1 \in y_2 + B_{t_1, r_2} \}, \\
\chi_2 &:= \mathbf{1} \{ d_B(x_2, B(y_1, r_1)) \leq t_2 \} = \mathbf{1} \{ x_2 \in y_1 + B_{t_2, r_1} \},
\end{aligned}$$

and

$$\tilde{\kappa}_B(x_1, x_2; t_1, t_2, R) := |(x_1 + B_{t_1, R}) \cap (x_2 + B_{t_2, R}) \cap F_o^c|_d.$$

Next, we fix $x_1 \in E_o$ and $x_2 \in E_z$, for the moment, and distinguish several cases.

1. If $y_1 \in F_o^c$ and $y_2 \in F_z^c$, then $t_1 \geq (m - r_1)^+/c_B$, $t_2 \geq (m - r_2)^+/c_B$ and using (3.12) we get

$$\begin{aligned} & \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| I_2(\mathbb{R}^d, \mathbb{R}^d) \right| \\ &= \left| (1 - \chi_1)(1 - \chi_2) \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d}(x_1, x_2; t_1, t_2) - \bar{F}_B(t_1) \bar{F}_B(t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) + \chi_1 + \chi_2. \end{aligned}$$

2. If $y_1 \in F_o$ and $y_2 \in F_z^c \cap F_o$, then $t_2 \geq (m - r_2)^+/c_B$ and using (3.13) we get

$$\begin{aligned} & \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) \right| \\ &= \left| (1 - \chi_1)(1 - \chi_2) \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right) (x_1, x_2; t_1, t_2) - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \bar{F}_B(t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; s_1, s_2, R) + \chi_1 + \chi_2. \end{aligned}$$

3. If $y_1 \in F_o$ and $y_2 \in F_z^c \cap F_o^c$, then $t_2 \geq (m - r_2)^+/c_B$ and using (3.13) we get

$$\begin{aligned} & \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) \right| \\ &= \left| (1 - \chi_2) \left((1 - \chi_1) \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right) (x_1, x_2; t_1, t_2) - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \bar{F}_B(t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; s_1, s_2, R) + \chi_1 + \chi_2. \end{aligned}$$

4. If $y_1 \in F_o^c \cap F_z$ and $y_2 \in F_z$, then $t_1 \geq (m - r_1)^+/c_B$ and using (3.14) we get

$$\begin{aligned} & \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) \right| \\ &= \left| (1 - \chi_1)(1 - \chi_2) \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} \right) (x_1, x_2; t_1, t_2) - \bar{F}_B(t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) + \chi_1 + \chi_2. \end{aligned}$$

5. If $y_1 \in F_o^c \cap F_z^c$ and $y_2 \in F_z$, then $t_1 \geq (m - r_1)^+/c_B$ and using (3.14) we get

$$\begin{aligned} & \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) \right| \\ &= \left| (1 - \chi_1) \left((1 - \chi_2) \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} \right) (x_1, x_2; t_1, t_2) - \bar{F}_B(t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\ &\leq \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) + \chi_1 + \chi_2. \end{aligned}$$

6. If $y_1 \in F_o \cap F_z$ and $y_2 \in F_z \cap F_o$, then by (3.9) and (3.15),

$$\begin{aligned}
& \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| \\
&= \left| (1 - \chi_1)(1 - \chi_2) \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \right. \\
&\quad \left. - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\
&\leq (\nu_1(x_1, t_1)\nu_2(x_2, t_2) + \nu_1(x_1, t_1) + \nu_2(x_2, t_2))\gamma\mathbb{E}\kappa_B(x_2 - x_1; t_1, t_2, R) \\
&\quad + \gamma\mathbb{E}\tilde{\kappa}_B(x_1, x_2; t_1, t_2, R) + (\chi_1 + \chi_2) \left(\nu_1(x_1, t_1)\nu_2(x_2, t_2) + \sqrt{\nu_1(x_1, t_1)\nu_2(x_2, t_2)} \right).
\end{aligned}$$

7. If $y_1 \in F_o \cap F_z^c$ and $y_2 \in F_z \cap F_o$, then by (3.6), (3.9) and (3.15),

$$\begin{aligned}
& \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| (1 - \chi_1)(1 - \chi_2) \right. \\
&\quad \times \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{F_o, \mathbb{R}^d} \right) (x_1, x_2; t_1, t_2) - (1 - \chi_1) \left(\bar{F}_B^{\mathbb{R}^d, F_z} - \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \\
&\quad \left. - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\
&\leq (\nu_1(x_1, t_1)\nu_2(x_2, t_2) + \nu_1(x_1, t_1) + \nu_2(x_2, t_2))\gamma\mathbb{E}\kappa_B(x_2 - x_1; t_1, t_2, R) \\
&\quad + \gamma\mathbb{E}\tilde{\kappa}_B(x_1, x_2; t_1, t_2, R) + \chi_1 \left(\nu_1(x_1, t_1)\nu_2(x_2, t_2) + \sqrt{\nu_1(x_1, t_1)\nu_2(x_2, t_2)} \right) \\
&\quad + \chi_2\nu_1(x_1, t_1).
\end{aligned}$$

8. If $y_1 \in F_o \cap F_z$ and $y_2 \in F_z \cap F_o^c$, then by (3.7), (3.9) and (3.15),

$$\begin{aligned}
& \left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| = \left| (1 - \chi_1)(1 - \chi_2) \right. \\
&\quad \times \left(\bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} - \bar{F}_B^{\mathbb{R}^d, F_z} \right) (x_1, x_2; t_1, t_2) - (1 - \chi_2) \left(\bar{F}_B^{F_o, \mathbb{R}^d} - \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \\
&\quad \left. - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\
&\leq (\nu_1(x_1, t_1)\nu_2(x_2, t_2) + \nu_1(x_1, t_1) + \nu_2(x_2, t_2))\gamma\mathbb{E}\kappa_B(x_2 - x_1; t_1, t_2, R) \\
&\quad + \gamma\mathbb{E}\tilde{\kappa}_B(x_1, x_2; t_1, t_2, R) + \chi_1\nu_2(x_2, t_2) \\
&\quad + \chi_2 \left(\nu_1(x_1, t_1)\nu_2(x_2, t_2) + \sqrt{\nu_1(x_1, t_1)\nu_2(x_2, t_2)} \right).
\end{aligned}$$

9. If $y_1 \in F_o \cap F_z^c$ and $y_2 \in F_z \cap F_o^c$, then $\chi_1 = 1$ implies $t_1 \geq (m - r_2)^+/c_B$ and $\chi_2 = 1$ implies

$t_2 \geq (m - r_1)^+/c_B$, and by (3.15) we have

$$\begin{aligned}
& |I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z)| = \left| \left((1 - \chi_1)(1 - \chi_2) \bar{F}_B^{\mathbb{R}^d, \mathbb{R}^d} \right. \right. \\
& \quad \left. \left. - (1 - \chi_2) \bar{F}_B^{F_o, \mathbb{R}^d} - (1 - \chi_1) \bar{F}_B^{\mathbb{R}^d, F_z} + \bar{F}_B^{F_o, F_z} \right) (x_1, x_2; t_1, t_2) \right. \\
& \quad \left. - \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_o} \right) (x_1; t_1) \left(\bar{F}_B^{\mathbb{R}^d} - \bar{F}_B^{F_z} \right) (x_2; t_2) \right| \\
& \leq (\nu_1(x_1, t_1) \nu_2(x_2, t_2) + \nu_1(x_1, t_1) + \nu_2(x_2, t_2)) \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) \\
& \quad + \gamma \mathbb{E} \tilde{\kappa}_B(x_1, x_2; t_1, t_2, R) + \chi_1 + \chi_2.
\end{aligned}$$

Altogether this gives

$$\left| I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_o, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_o, F_z) \right| \leq I_2^*(x_1, x_2, y_1, y_2, r_1, r_2),$$

where

$$\begin{aligned}
I_2^*(x_1, x_2, y_1, y_2, r_1, r_2) &:= (\psi(t_1, r_1) \psi(t_2, r_2) + 2\psi(t_2, r_2) + 2\psi(t_1, r_1)) \\
&\quad \times (\gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) + \chi_1 + \chi_2) \\
&\quad + 4(\nu(t_1) \nu(t_2) + \nu(t_1) + \nu(t_2)) \gamma \mathbb{E} \kappa_B(x_2 - x_1; t_1, t_2, R) + 4\gamma \mathbb{E} \tilde{\kappa}_B(x_1, x_2; t_1, t_2, R) \\
&\quad + 2(\chi_1 + \chi_2) \left(\nu(t_1) \nu(t_2) + \sqrt{\nu(t_1) \nu(t_2)} \right) \\
&\quad + \chi_2 \nu(t_1) + \chi_1 \nu(t_2) + \chi_1 \psi(t_1, r_2) + \chi_2 \psi(t_2, r_1)
\end{aligned}$$

does not depend on z . We recall the dependence of χ_1, χ_2 on x_1, x_2, y_1, y_2 and then carry out the substitutions $x_1 - y_1 \rightarrow x_1$ and $x_2 - y_2 \rightarrow x_2$ to get

$$\begin{aligned}
S_2 &\leq \sum_{z \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, r_1 B^d), r_1) g(d_B(x_2, r_2 B^d), r_2) \mathbf{1}\{x_1 + y_1 \in E_o\} \\
&\quad \times \mathbf{1}\{x_2 + y_2 \in E_z\} I_2^*(x_1 + y_1, x_2 + y_2, y_1, y_2, r_1, r_2) dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2) dx_2 dx_1 \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, r_1 B^d), r_1) g(d_B(x_2, r_2 B^d), r_2) \mathbf{1}\{x_1 + y_1 \in E_o\} \\
&\quad \times I_2^*(x_1 + y_1, x_2 + y_2, y_1, y_2, r_1, r_2) dy_1 dy_2 \mathbb{G}(dr_1) \mathbb{G}(dr_2) dx_2 dx_1.
\end{aligned}$$

To the inner integrals we apply the relations

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x_1 + y_1 \in E_o\} \chi_1 dy_2 dy_1 = \int_{E_o - x_1} \int_{\mathbb{R}^d} \mathbf{1}\{x_1 + y_1 - y_2 \in B_{t_1, r_2}\} dy_2 dy_1 = |B_{t_1, r_2}|_d, \\
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x_1 + y_1 \in E_o\} \chi_2 dy_2 dy_1 = \int_{E_o - x_1} \int_{\mathbb{R}^d} \mathbf{1}\{x_2 + y_2 - y_1 \in B_{t_2, r_1}\} dy_2 dy_1 = |B_{t_2, r_1}|_d, \\
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x_1 + y_1 \in E_o\} \mathbb{E} \kappa_B(x_2 + y_2 - x_1 - y_1; t_1, t_2, R) dy_2 dy_1 = \mathbb{E} |B_{t_1, R}|_d |B_{t_2, R}|_d, \\
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x_1 + y_1 \in E_o\} \mathbb{E} \tilde{\kappa}_B(x_1 + y_1, x_2 + y_2; t_1, t_2, R) dy_2 dy_1 \\
&= \int_{\mathbb{R}^d} \mathbf{1}\{x_1 + y_1 \in E_o\} \mathbb{E} |(x_1 + y_1 + B_{t_1, R}) \cap F_o^c|_d |B_{t_2, R}|_d \leq \mathbb{E} |(E_o \oplus B_{t_1, R}) \cap F_o^c|_d |B_{t_2, R}|_d.
\end{aligned}$$

In the last two equations we used [20, Theorem 5.2.1]. Consequently, (2.12) and (2.11) yield

$$\begin{aligned}
S_2 \leq & \int_C \int_C \int_0^\infty \int_0^\infty f(t_1) f(t_2) \left[(\psi(t_1, r_1) \psi(t_2, r_2) + 2\psi(t_2, r_2) + 2\psi(t_1, r_1)) \right. \\
& \times (\gamma \mathbb{E} |B_{t_1, R}|_d |B_{t_2, R}|_d + |B_{t_1, r_2}|_d + |B_{t_2, r_1}|_d) \\
& + 4(\nu(t_1) \nu(t_2) + \nu(t_1) + \nu(t_2)) \gamma \mathbb{E} |B_{t_1, R}|_d |B_{t_2, R}|_d + 4\gamma \mathbb{E} |(E_o \oplus B_{t_1, R}) \cap F_o^c|_d |B_{t_2, R}|_d \\
& + 2(|B_{t_1, r_2}|_d + |B_{t_2, r_1}|_d) \left(\nu(t_1) \nu(t_2) + \sqrt{\nu(t_1) \nu(t_2)} \right) + |B_{t_2, r_1}|_d \nu(t_1) + |B_{t_1, r_2}|_d \nu(t_2) \\
& \left. + |B_{t_1, r_2}|_d \psi(t_1, r_2) + |B_{t_2, r_1}|_d \psi(t_2, r_1) \right] dt_2 dt_1 \mathbb{G}(dr_2) \mathbb{G}(dr_1).
\end{aligned}$$

Using $\mathbb{E} |B_{t, R}|_d \leq \mathbb{E} |(E_o \oplus B_{t, R})|_d \leq c_1(1+t^d)$, cf. (3.11), the Cauchy-Schwarz inequality and assumption (3.1), it follows from the Lebesgue dominated convergence theorem that $S_2 \rightarrow 0$ as $m \rightarrow \infty$. \square

Now we are dealing with the asymptotic normality of $\widehat{\mathbb{G}}_n(C)$.

Theorem 3.2. *Assume that (3.1) is satisfied. Let $W_n = [-n, n]^d$. If $C \subset \mathbb{R}^+$ is a Borel set, then*

$$\sqrt{|W_n|_d} \left(\widehat{\mathbb{G}}_n(C) - \mathbb{G}(C) \right) \xrightarrow[n \rightarrow \infty]{d} N(0, \sigma_{\mathbb{G}}^2(C)),$$

where

$$\sigma_{\mathbb{G}}^2(C) := \frac{1}{\gamma^2 \beta^2} [(1 - \mathbb{G}(C)) \sigma^2(C) + \mathbb{G}(C) \sigma^2(\mathbb{R}^+ \setminus C) - \mathbb{G}(C)(1 - \mathbb{G}(C)) \sigma^2(\mathbb{R}^+)] \quad (3.16)$$

and $\sigma^2(\cdot)$ is given by (2.14). If $0 < \mathbb{G}(C) < 1$, then $\sigma_{\mathbb{G}}^2(C) > 0$.

Proof. Using (2.17) and Slutsky's theorem, the weak limit of $\sqrt{|W_n|_d} \left(\widehat{\mathbb{G}}_n(C) - \mathbb{G}(C) \right)$ coincides with the weak limit of

$$Y_n := \frac{1}{\gamma \beta \sqrt{|W_n|_d}} (\eta_{W_n}(C) - \eta_{W_n}(\mathbb{R}^+) \mathbb{G}(C)).$$

Observing that

$$\gamma \beta Y_n = \frac{1}{\sqrt{|W_n|_d}} \int_{W_n} (\mathbf{1}_{\{r_B(x, Z) \in C\}} - \mathbb{G}(C)) f(d_B(x, Z)) h_B(d_B(x, Z), r_B(x, Z))^{-1} dx,$$

we can proceed along the same lines as in the proof of Theorem 3.1 and obtain

$$\gamma \beta Y_n \xrightarrow[n \rightarrow \infty]{d} N(0, \gamma^2 \beta^2 \sigma_{\mathbb{G}}^2(C)),$$

provided we can identify the asymptotic variance $\sigma_{\mathbb{G}}^2(C)$ of Y_n . Theorem 2.4 implies that

$$\begin{aligned}
& \gamma^2 \beta^2 \lim_{n \rightarrow \infty} \text{Var } Y_n \\
& = \sigma^2(C) + \mathbb{G}(C)^2 \sigma^2(\mathbb{R}^+) - 2\mathbb{G}(C) \lim_{n \rightarrow \infty} \frac{1}{|W_n|_d} \text{Cov}(\eta_{W_n}(C), \eta_{W_n}(\mathbb{R}^+)). \quad (3.17)
\end{aligned}$$

Since $\eta_{W_n}(\cdot)$ is additive, we obtain from Theorem 2.4 that

$$\begin{aligned} 2 \lim_{n \rightarrow \infty} \frac{1}{|W_n|_d} \text{Cov}(\eta_{W_n}(C), \eta_{W_n}(\mathbb{R}^+)) &= 2\sigma^2(C) + 2 \lim_{n \rightarrow \infty} \frac{1}{|W_n|_d} \text{Cov}(\eta_{W_n}(C), \eta_{W_n}(\mathbb{R}^+ \setminus C)) \\ &= 2\sigma^2(C) + \lim_{n \rightarrow \infty} \frac{1}{|W_n|_d} (\text{Var } \eta_{W_n}(\mathbb{R}^+) - \text{Var } \eta_{W_n}(C) - \text{Var } \eta_{W_n}(\mathbb{R}^+ \setminus C)) \\ &= 2\sigma^2(C) + \sigma^2(\mathbb{R}^+) - \sigma^2(C) - \sigma^2(\mathbb{R}^+ \setminus C) = \sigma^2(C) + \sigma^2(\mathbb{R}^+) - \sigma^2(\mathbb{R}^+ \setminus C). \end{aligned}$$

Inserting this result into (3.17) we obtain (3.16) upon some simplification.

To prove the last assertion, we define $\tilde{g}(t, s) := (\mathbf{1}\{s \in C\} - \mathbb{G}(C))f(t)h_B(t, s)^{-1}$ and assume that $0 < \mathbb{G}(C) < 1$. For a convex body $W \subset \mathbb{R}^d$ we need to consider the variance of

$$H_W := \int_W \tilde{g}(d_B(x, Z), r_B(x, Z)) \, dx.$$

As in the proof of the positivity assertion in Theorem 2.4 we obtain that

$$\text{Var } H_W \geq \gamma \int_0^\infty \int_{\mathbb{R}^d} \tilde{h}(y, r)^2 \, dy \, \mathbb{G}(dr), \quad (3.18)$$

where

$$\begin{aligned} \tilde{h}(y, r) &:= \mathbb{E} \int_W \mathbf{1}\{d_B(o, B(y-x, r)) < d_B(o, Z)\} \tilde{g}(d_B(o, B(y-x, r)), r) \, dx \\ &\quad - \mathbb{E} \int_W \mathbf{1}\{d_B(o, B(y-x, r)) < d_B(o, Z)\} \tilde{g}(d_B(o, Z), r_B(o, Z)) \, dx. \end{aligned}$$

By (1.4) and the definition of \tilde{g} the second expectation on the above right-hand side vanishes for all $y \in W$ and $r \geq 0$. Therefore

$$\tilde{h}(y, r) = \int_W \bar{F}_B(d_B(o, B(y-x, r))) \tilde{g}(d_B(o, B(y-x, r)), r) \, dx.$$

Again as in the proof of Theorem 2.4 we let $C' := \mathbb{R}^+ \setminus C$ and obtain from Jensen's inequality and (3.18) that

$$\begin{aligned} \frac{\sqrt{\text{Var } H_W}}{\sqrt{|W|_d}} &\geq \frac{c}{|W|_d} \int_{C'} \int_W \int_W \bar{F}_B(d_B(o, B(y-x, r))) f(d_B(o, B(y-x, r))) \\ &\quad \times (h_B(d_B(o, B(y-x, r)), r))^{-1} \, dx \, dy \, \mathbb{G}(dr) \\ &= \frac{c}{|W|_d} \int_{C'} \int_{\mathbb{R}^d} |W \cap (W-y)| \bar{F}_B(d_B(o, B(y, r))) f(d_B(o, B(y, r))) \\ &\quad \times (h_B(d_B(o, B(y, r)), r))^{-1} \, dy \, \mathbb{G}(dr), \end{aligned}$$

where $c > 0$ is a constant not depending on W . Hence it is sufficient to show that

$$\int_{C'} \int_{\mathbb{R}^d} \bar{F}_B(d_B(o, B(y, r))) f(d_B(o, B(y, r))) (h_B(d_B(o, B(y, r)), r))^{-1} \, dy \, \mathbb{G}(dr) > 0.$$

By (2.12) the above integral equals $\mathbb{G}(C') \int_{\mathbb{R}^d} \bar{F}_B(t) f(t) \, dt$, which is positive by (1.6). \square

Remark 3.3. After some manipulation we get

$$\gamma^2 \beta^2 \sigma_{\mathbb{G}}^2(C) = \gamma \int_{\mathbb{R}^d} \tilde{\tau}_1(C, u) \, du + \gamma^2 \int_{\mathbb{R}^d} \tilde{\tau}_2(C, u) \, du,$$

where

$$\begin{aligned} \tilde{\tau}_1(C, u) := & \int_0^\infty \int_{\mathbb{R}^d} \frac{f(d_B(o, B(x, r)))}{h_B(d_B(o, B(x, r)), r)} \frac{f(d_B(u, B(x, r)))}{h_B(d_B(u, B(x, r)), r)} \\ & \times \bar{F}_B^{(2)}(u; d_B(o, B(x, r)), d_B(u, B(x, r))) (\mathbf{1}\{r \in C\} - \mathbb{G}(C))^2 \, dx \, \mathbb{G}(dr) \end{aligned}$$

and

$$\begin{aligned} \tilde{\tau}_2(C, u) := & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_2, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\ & \times \mathbf{1}\{d_B(x_2, B(u, r_2)) \leq d_B(x_1, B(o, r_1))\} \mathbf{1}\{d_B(x_1, B(-u, r_1)) \leq d_B(x_2, B(o, r_2))\} \\ & \times \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) (\mathbf{1}\{r_1 \in C\} - \mathbb{G}(C)) (\mathbf{1}\{r_2 \in C\} - \mathbb{G}(C)) \\ & \times \, dx_1 \, dx_2 \, \mathbb{G}(dr_1) \, \mathbb{G}(dr_2). \end{aligned}$$

This relation can be also obtained directly by an analogue of the proof of Theorem 2.4.

4 The planar case

We mentioned at the beginning that the estimator $\hat{\mathbb{G}}$ which we discussed so far is based on the data $\{(d_B(x, Z), r_B(x, Z)) : x \in W \setminus Z\}$ and therefore may require information from outside the window W . To overcome this problem, a common procedure in spatial statistics is the so-called *minus sampling*, which can be used, e.g., if the radius distribution \mathbb{G} is concentrated on an interval $[0, r_0]$, $0 < r_0 < \infty$. We can avoid such a condition by assuming that the function f is concentrated on an interval $[0, \varepsilon]$ with $\varepsilon > 0$. If we then assume that Z is observable in a window $W^{(\varepsilon)}$ which contains $W \oplus \varepsilon B$, then, for each $x \in W$, we have either $f(d_B(x, Z)) = 0$ or $d_B(x, Z) \leq \varepsilon$, in which case the (almost surely unique) contact point $(x + d_B(x, Z)B) \cap Z$ lies in $W^{(\varepsilon)}$.

For practical applications, the planar case $d = 2$ is particularly important. Also, then, the spherical case $B = B^2$ and the linear case $B = [0, u]$ (with a given direction u) play a major role. For simplicity, in the following considerations we concentrate on the window $W = [0, 1]^2$ and we assume, as explained above, that f is concentrated on $[0, \varepsilon]$, $\varepsilon > 0$, and that Z is observed in $W^{(\varepsilon)}$. Let $\tilde{C}_1, \dots, \tilde{C}_k$ be the (connected and relatively open) visible arcs in $\partial Z \cap W^{(\varepsilon)}$. We need not know whether some of these arcs belong to the same particle. By $C_1 \subset \tilde{C}_1, \dots, C_k \subset \tilde{C}_k$ we denote the corresponding “effective” arcs; these consist of the points $(x + d_B(x, Z)B) \cap Z \in \tilde{C}_i$, $x \in W \setminus Z$, for which $d_B(x, Z) \leq \varepsilon$. Let r_i be the radius and l_i the length of C_i , and let A_i be the set of points $x \in W \setminus Z$ with $d_B(x, Z) \leq \varepsilon$ and which project onto C_i , $i = 1, \dots, k$, in the sense that $(x + d_B(x, Z)B) \cap Z$ consists of a unique point and this point lies in C_i , for $x \in A_i$. Then our estimator $\hat{\mathbb{G}}$ is of the form

$$\hat{\mathbb{G}} = \frac{1}{\sum_{i=1}^k w_i} \sum_{i=1}^k w_i \delta_{r_i},$$

where the weight w_i is given by

$$w_i = \int_{A_i} f(d_B(x, Z)) h_B(d_B(x, Z), r_B(x, Z))^{-1} dx.$$

For $B = B^2$ we have $h_{B^2}(t, r) = 2\pi(t + r)$ (see Remark 2.3), hence if $f(t) = \varepsilon^{-1} \mathbf{1}\{t \leq \varepsilon\}$ then

$$w_i = \frac{1}{2\pi\varepsilon} \int_{A_i \cap (C_i + \varepsilon B^2)} \frac{1}{d_{B^2}(x, C_i) + r_i} dx.$$

If we let $\varepsilon \rightarrow 0$, the weights converge to $w_i = l_i/(2\pi r_i)$ if $r_i > 0$ and to $w_i = 1$ if $r_i = l_i = 0$. Then the estimator becomes

$$\widehat{\mathbb{G}}_o := \left(\sum_{i=1}^k \frac{l_i}{r_i} \right)^{-1} \sum_{i=1}^k \frac{l_i}{r_i} \delta_{r_i}$$

with l_i/r_i interpreted as 2π if $r_i = l_i = 0$. Notice also that the outer sampling window $W^{(\varepsilon)}$ then shrinks to W , so that in the limit only information in W is needed. The estimator $\widehat{\mathbb{G}}_o$ was discussed by Hall [3, Chapter 5.6] (more generally, he considered estimators of $\mathbb{E}A(R)$, for a given function A ; $\widehat{\mathbb{G}}_o$ corresponds to the case $A = \mathbf{1}_C$).

For $B = [0, u]$ (with $u \in \{\pm e_1, \pm e_2\}$), assuming (in the linear case) that $\mathbb{G}(\{0\}) = 0$ and hence $r_i > 0$, and again choosing $f(t) = \varepsilon^{-1} \mathbf{1}\{t \leq \varepsilon\}$, we get $h_B(t, r) = 2r$ and

$$w_i = \frac{1}{2\varepsilon r_i} \int_{A_i \cap (C_i + \varepsilon[0, -u])} dx.$$

This yields an estimator $\widehat{\mathbb{G}}_{l,u}$ in the limit $\varepsilon \rightarrow 0$, which is given by

$$\widehat{\mathbb{G}}_{l,u} := \left(\sum_{i=1}^k \frac{l_i(u)}{r_i} \right)^{-1} \sum_{i=1}^k \frac{l_i(u)}{r_i} \delta_{r_i}.$$

Here, $l_i(u)$ is the length of the projection of the visible part of C_i in direction u (projected onto the line orthogonal to u). The estimator can be improved by combining $u = e_1, -e_1, e_2, -e_2$,

$$\widehat{\mathbb{G}}_l := \frac{1}{4} \left(\widehat{\mathbb{G}}_{l,e_1} + \widehat{\mathbb{G}}_{l,-e_1} + \widehat{\mathbb{G}}_{l,e_2} + \widehat{\mathbb{G}}_{l,-e_2} \right).$$

For applications, it would be natural to choose $\varepsilon = 1$ which yields weights

$$w_i = \frac{|A_i|_2}{2r_i}, \quad i = 1, \dots, k,$$

and gives the estimator

$$\widehat{\mathbb{G}} = \left(\sum_{i=1}^k \frac{|A_i|_2}{r_i} \right)^{-1} \sum_{i=1}^k \frac{|A_i|_2}{r_i} \delta_{r_i}.$$

Hence, in this case and with $u = e_1$, information in $[0, 2] \times [0, 1]$ would be required and the estimation is based on the areas of the regions $A_i \subset [0, 1]^2$. Of course, the estimation can be again improved by combining the estimators for $u = e_1, -e_1, e_2, -e_2$ which are available if Z is observed in $[-1, 2]^2$.

If we do not have information from outside W , then we may use a minus sampling approach and replace W by the eroded window $W_{\ominus\varepsilon} := \{x \in W : x + \varepsilon B \subset W\}$, i.e. we consider the following estimator

$$\widehat{\mathbb{G}}_{\ominus\varepsilon}(C) := \frac{\eta_{W_{\ominus\varepsilon}}(C)}{\eta_{W_{\ominus\varepsilon}}(\mathbb{R}^+)}.$$

Another possibility would be to use the naive approach which ignores edge effects. Then we have the *uncorrected estimator*

$$\widehat{\mathbb{G}}_u(C) := \frac{\eta_{W,u}(C)}{\eta_{W,u}(\mathbb{R}^+)},$$

where

$$\eta_{W,u}(C) := \int_W \mathbf{1}\{r_B(x, Z \cap W) \in C\} \frac{f(d_B(x, Z \cap W))}{h_B(d_B(x, Z \cap W), r_B(x, Z \cap W))} dx.$$

If $B = [0, u]$, then it can happen that $d_B(x, Z \cap W) = \infty$. In that case we use our convention concerning $f \cdot h_B^{-1}$, i.e. the points x satisfying $d_B(x, Z \cap W) = \infty$ do not contribute to $\eta_{W,u}(C)$. Besides minus sampling there exist more sophisticated methods of edge correction in the statistics of spatial point processes. We adopt the idea of local minus sampling that was originally applied in [4] to the estimation of the nearest neighbour distance distribution function for stationary point processes (see also [5]). We use only points that are closer to Z than to the boundary of the window W . This gives the *Hanisch type estimator*

$$\widehat{\mathbb{G}}_H(C) := \frac{\eta_{W,H}(C)}{\eta_{W,H}(\mathbb{R}^+)},$$

where

$$\eta_{W,H}(C) := \int_W \mathbf{1}\{r_B(x, Z) \in C\} \mathbf{1}\{d_B(x, Z) \leq d_B(x, \partial W)\} \frac{f(d_B(x, Z))}{h_B(d_B(x, Z), r_B(x, Z))} dx.$$

Note that for $B = [0, u]$ the estimators $\widehat{\mathbb{G}}_H$ and $\widehat{\mathbb{G}}_u$ coincide.

In practical applications one has to replace in (1.5) the integration with respect to Lebesgue measure by an integration with respect to a discrete measure. This still gives a ratio-unbiased estimator of \mathbb{G} .

We compare the performance of the different estimators discussed above through computer simulations. We simulate a stationary planar Boolean model with spherical grains, given by (1.1). The observation window W is the unit square $[0, 1]^2$. The distribution \mathbb{G} is assumed to be uniform on $(0.05, 0.1)$. We approximate the integrals over W by Riemannian sums over a rectangular grid of points $L_h \cap W$, where

$$L_h := \{(k - 1/2)h, (l - 1/2)h) : k, l \in \mathbb{N}\}.$$

For our purposes, we choose $h = 1/300$.

We take $f(t) = \varepsilon^{-1} \mathbf{1}\{t \leq \varepsilon\}$ for different choices of ε and compare the estimator $\widehat{\mathbb{G}}$, given by (1.8), with the estimators $\widehat{\mathbb{G}}_\circ$ (for spherical B) and $\widehat{\mathbb{G}}_l$ (for linear B) corresponding to the limiting case $\varepsilon \rightarrow 0$. The estimators $\widehat{\mathbb{G}}_{\ominus\varepsilon}$, $\widehat{\mathbb{G}}_u$ and $\widehat{\mathbb{G}}_H$ are also evaluated. For linear $B = [0, u]$ we always combine the corresponding estimators for $u = e_1, -e_1, e_2, -e_2$, this leads to a noticeable improvement.

The radius distribution \mathbb{G} is uniquely determined by the distribution function $G(t) = \mathbb{G}([0, t])$, $t \geq 0$. We measure the quality of the estimators by the Kolmogorov-Smirnov distance

$$d_{KS}(\widehat{G}, G) := \sup_{s \geq 0} |\widehat{G}(s) - G(s)|$$

and the Cramér-von Mises distance

$$d_{CvM}(\widehat{G}, G) := \int_{0.05}^{0.1} (\widehat{G}(s) - G(s))^2 \frac{ds}{0.05}.$$

We have generated 100 independent realizations of the Boolean model Z with chosen intensity γ . For each realization we have determined several estimators under study. The sample means of corresponding Kolmogorov-Smirnov and Cramér-von Mises distances over 100 simulations are demonstrated in Table 1 for $\gamma = 25$ and in Table 2 for $\gamma = 100$. The results show that smaller values of ε are more

estimator	d_{KS}		$1000 \cdot d_{\text{CvM}}$	
	spherical B	linear B	spherical B	linear B
$\hat{\mathbb{G}}, \varepsilon = 1$	0.178	0.147	7.921	5.139
$\hat{\mathbb{G}}, \varepsilon = 0.05$	0.172	0.170	7.317	7.101
$\hat{\mathbb{G}}, \varepsilon = 0.01$	0.172	0.172	7.295	7.292
$\hat{\mathbb{G}}_o$ or $\hat{\mathbb{G}}_l$	0.171	0.172	7.243	7.257
$\hat{\mathbb{G}}_{\ominus \varepsilon}, \varepsilon = 0.05$	0.191	0.177	9.243	7.753
$\hat{\mathbb{G}}_{\ominus \varepsilon}, \varepsilon = 0.01$	0.176	0.173	7.674	7.435
$\hat{\mathbb{G}}_u, \varepsilon = 1$	0.182	0.179	8.389	7.890
$\hat{\mathbb{G}}_u, \varepsilon = 0.05$	0.173	0.169	7.480	7.553
$\hat{\mathbb{G}}_u, \varepsilon = 0.01$	0.173	0.168	7.322	7.472
$\hat{\mathbb{G}}_H, \varepsilon = 1$	0.187	0.179	9.003	7.890
$\hat{\mathbb{G}}_H, \varepsilon = 0.05$	0.179	0.169	8.023	7.553
$\hat{\mathbb{G}}_H, \varepsilon = 0.01$	0.174	0.168	7.462	7.472

Table 1: Sample means of distances between distribution functions computed from 100 realizations of a Boolean model with intensity $\gamma = 25$ and uniform radius distribution on $(0.05, 0.1)$.

preferable. The limiting estimators $\hat{\mathbb{G}}_o$ and $\hat{\mathbb{G}}_l$ produced the smallest error. They are outperformed only in the case of smaller intensity and linear B where our estimator, given by (1.8), with larger ε , gives better results. However, this estimator uses also information from outside W . Simulation studies for exponentially distributed radii (not presented) show very similar results. A change of resolution h has only a minor influence on the quality of the estimators. For intensity $\gamma \gg 100$ the deviation from the radius distribution increases which is intuitively clear because many balls are covered so that their radii are not available for the estimators.

References

- [1] Ballani, F. On second-order characteristics of germ-grain models with convex grains, *Mathematika* **53** (2006), 255–285.
- [2] Billingsley, P. *Convergence of Probability Measures*, 2nd edition, John Wiley & Sons, New York, 1999.
- [3] Hall, P. *Introduction to the Theory of Coverage Processes*, John Wiley & Sons, New York, 1988.
- [4] Hanisch, K.-H. Some remarks on estimators of the distribution function of nearest neighbour distance in stationary spatial point patterns, *Statistics* **15** (1984), 409–412.
- [5] Hansen, M. B., Baddeley, A. J., Gill, R. D. First contact distributions for spatial patterns: regularity and estimation, *Adv. Appl. Prob. (SGSA)* **31** (1999), 15–33.
- [6] Heinrich, L. Asymptotic methods in statistics of random point processes, In: Spodarev, E. (ed) *Stochastic Geometry, Spatial Statistics and Random Fields*, pp. 115–150, Lecture Notes in Mathematics **2068**. Springer, Berlin, 2013.
- [7] Heinrich, L., Molchanov, I. S. Central limit theorem for a class of random measures associated with germ-grain models, *Adv. Appl. Prob. (SGSA)* **31** (1999), 283–314.

estimator	d_{KS}		$1000 \cdot d_{CvM}$	
	spherical B	linear B	spherical B	linear B
$\widehat{G}, \varepsilon = 1$	0.147	0.134	5.506	4.406
$\widehat{G}, \varepsilon = 0.05$	0.145	0.131	5.294	4.238
$\widehat{G}, \varepsilon = 0.01$	0.132	0.128	4.276	4.008
\widehat{G}_o or \widehat{G}_l	0.127	0.127	3.919	3.928
$\widehat{G}_{\ominus \varepsilon}, \varepsilon = 0.05$	0.158	0.135	6.162	4.460
$\widehat{G}_{\ominus \varepsilon}, \varepsilon = 0.01$	0.134	0.129	4.359	4.029
$\widehat{G}_u, \varepsilon = 1$	0.150	0.140	5.710	4.838
$\widehat{G}_u, \varepsilon = 0.05$	0.147	0.137	5.431	4.807
$\widehat{G}_u, \varepsilon = 0.01$	0.133	0.129	4.299	4.208
$\widehat{G}_H, \varepsilon = 1$	0.150	0.140	5.602	4.838
$\widehat{G}_H, \varepsilon = 0.05$	0.148	0.137	5.438	4.807
$\widehat{G}_H, \varepsilon = 0.01$	0.133	0.129	4.323	4.208

Table 2: Sample means of distances between distribution functions computed from 100 realizations of a Boolean model with intensity $\gamma = 100$ and uniform radius distribution on $(0.05, 0.1)$.

- [8] Heinrich, L., Werner, W. Kernel estimation of the diameter distribution in Boolean models with spherical grains, *J. Nonparametr. Statist.* **12** (2000), 147–176.
- [9] Hug, D., Last, G. On support measures in Minkowski spaces and contact distributions in stochastic geometry, *Ann. Probab.* **28** (2000), 796–850.
- [10] Hug, D., Last, G., Weil, W. Generalized contact distributions of inhomogeneous Boolean models, *Adv. Appl. Prob. (SGSA)* **34** (2002), 21–47.
- [11] Kallenberg, O. *Foundations of Modern Probability*, 2nd edition, Springer-Verlag, New York, 2002.
- [12] Kiderlen, M., Weil, W. Measure-valued valuations and mixed curvature measures of convex bodies, *Geom. Dedicata* **76** (1999), 291–329.
- [13] Last, G., Penrose, M. D. Fock space representation, chaos expansion and covariance inequalities for general Poisson processes, *Prob. Theory Rel. Fields* **150** (2011), 663–690.
- [14] Molchanov, I. S. Estimation of the size distribution of spherical grains in the Boolean model, *Biometrical J.* **32** (1990), 877–886.
- [15] Molchanov, I. S. *Statistics of the Boolean Model for Practitioners and Mathematicians*, Wiley, Chichester, 1997.
- [16] Molchanov, I. S., Stoyan, D. Asymptotic properties of estimators for parameters of the Boolean model, *Adv. Appl. Prob.* **27** (1994), 63–86.
- [17] Rosén, B. A note on asymptotic normality of sums of higher-dimensionally indexed random variables, *Ark. Mat.* **8** (1969), 33–43.
- [18] Schneider, R. *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.
- [19] Stoyan, D., Kendall, W. S., Mecke, J. *Stochastic Geometry and its Applications*, 2nd edition, Wiley, Chichester, 1995.

[20] Schneider, R., Weil, W. *Stochastic and Integral Geometry*, Springer-Verlag, Berlin, 2008.

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